ZÁPADOČESKÁ UNIVERZITA V PLZNI

CAM
Centre of Applied Mathematics

# The beam operator and the Fučík spectrum

Gabriela Holubová & Petr Nečesal

Centre of Applied Mathematics, Department of Mathematics
Faculty of Applied Sciences, University of West Bohemia

EQUADIFF 11

# The Fučík Spectrum

$\Sigma(L) := \{(\alpha, \beta) \in \mathbb{R}^2 : \ Lu = \alpha u^+ - \beta u^-$ has a nontrivial solution$\}$

# The Fučík Spectrum

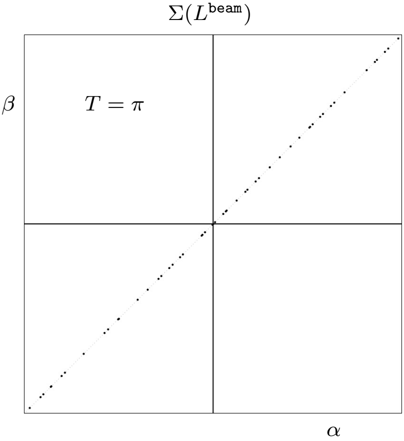
$$\Sigma(L) := \left\{(\alpha, \beta) \in \mathbb{R}^2 : \; Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\right\}$$

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t) & \text{on } (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 & \text{on } \mathbb{R}, \\ u(x,t) = u(x,t+T) & \text{on } (0,\pi) \times \mathbb{R}. \end{cases}$$

# The Fučík Spectrum

$$\Sigma(L) := \left\{(\alpha,\beta)\in\mathbb{R}^2 :\ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\right\}$$

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t) & \text{on } (0,\pi)\times\mathbb{R}, \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 & \text{on } \mathbb{R}, \\ u(x,t) = u(x,t+T) & \text{on } (0,\pi)\times\mathbb{R}. \end{cases}$$




$$\lambda_{k,l} = (2\pi/T)^2 l^2 - k^4, \qquad k\in\mathbb{N},\ l\in\mathbb{N}_0,$$

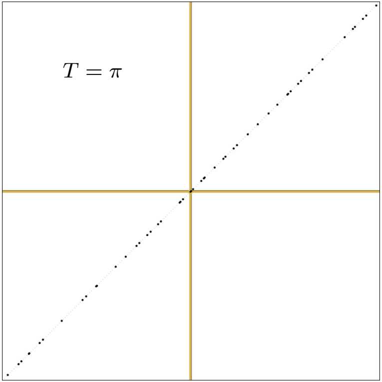
$$\Lambda := \left[\begin{array}{c|ccccc} \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \cdots \\ \hline \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \cdots \\ \lambda_{3,0} & \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \\ \lambda_{4,0} & \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \\ \vdots & \vdots & & & & \ddots \end{array}\right].$$

# The Fučík Spectrum

$$\Sigma(L) := \left\{(\alpha, \beta) \in \mathbb{R}^2 : \ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\right\}$$

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t) & \text{on } (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 & \text{on } \mathbb{R}, \\ u(x,t) = u(x,t+T) & \text{on } (0,\pi) \times \mathbb{R}. \end{cases}$$

$\Sigma(L^{\mathtt{beam}})$



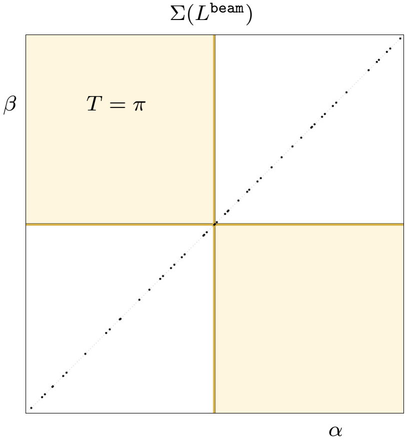

$$\lambda_{k,l} = (2\pi/T)^2 l^2 - k^4, \qquad k \in \mathbb{N},\ l \in \mathbb{N}_0,$$

$$\Lambda := \left[\begin{array}{c|ccccc} \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \cdots \\ \hline \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \cdots \\ \lambda_{3,0} & \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \\ \lambda_{4,0} & \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \\ \vdots & \vdots & & & & \ddots \end{array}\right].$$

# The Fučík Spectrum

$$\Sigma(L) := \left\{(\alpha,\beta) \in \mathbb{R}^2 :\ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\right\}$$

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t) & \text{on } (0,\pi)\times\mathbb{R}, \\ u\,(0,t) = u\,(\pi,t) = u_{xx}\,(0,t) = u_{xx}\,(\pi,t) = 0 \text{ on } \mathbb{R}, & \\ u(x,t) = u(x,t+T) & \text{on } (0,\pi)\times\mathbb{R}. \end{cases}$$




$$\lambda_{k,l} = (2\pi/T)^2 l^2 - k^4, \qquad k \in \mathbb{N},\ l \in \mathbb{N}_0,$$

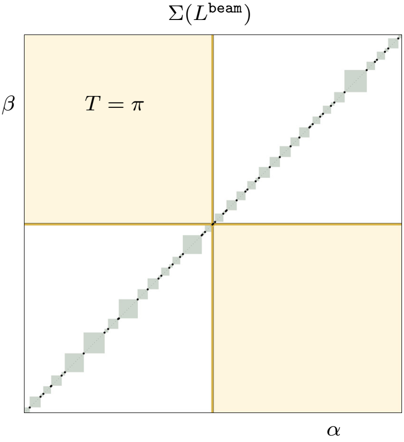
$$\Lambda := \left[\begin{array}{c|cccccc} \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \cdots \\ \hline \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \cdots \\ \lambda_{3,0} & \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \\ \lambda_{4,0} & \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \\ \vdots & \vdots & & & & \ddots \end{array}\right].$$

- inadmissible areas

# The Fučík Spectrum

$$\Sigma(L) := \left\{(\alpha,\beta)\in\mathbb{R}^2 :\ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\right\}$$

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t) & \text{on } (0,\pi)\times\mathbb{R}, \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 \text{ on } \mathbb{R}, & \\ u(x,t) = u(x,t+T) & \text{on } (0,\pi)\times\mathbb{R}. \end{cases}$$




$$\lambda_{k,l} = (2\pi/T)^2 l^2 - k^4, \qquad k\in\mathbb{N},\ l\in\mathbb{N}_0,$$

$$\Lambda := \left[\begin{array}{c|cccccc} \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \cdots \\ \hline \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \cdots \\ \lambda_{3,0} & \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \\ \lambda_{4,0} & \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \\ \vdots & \vdots & & & & \ddots \end{array}\right].$$

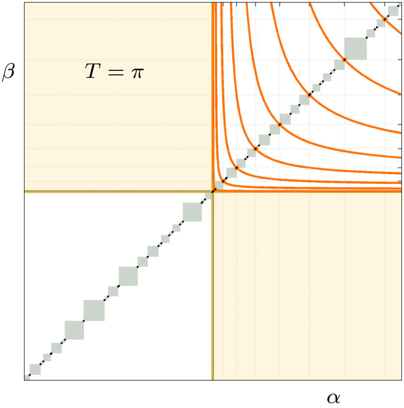
- inadmissible areas

# The Fučík Spectrum

$$\Sigma(L) := \{(\alpha,\beta)\in\mathbb{R}^2 :\ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\}$$

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t) & \text{on } (0,\pi)\times\mathbb{R}, \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 \text{ on } \mathbb{R}, & \\ u(x,t) = u(x,t+T) & \text{on } (0,\pi)\times\mathbb{R}. \end{cases}$$

$\Sigma(L^{\mathtt{beam}})$




$$\lambda_{k,l} = (2\pi/T)^2 l^2 - k^4, \qquad k\in\mathbb{N},\ l\in\mathbb{N}_0,$$

$$\Lambda := \left[\begin{array}{c|ccccc} \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \cdots \\ \hline \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \cdots \\ \lambda_{3,0} & \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \\ \lambda_{4,0} & \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \\ \vdots & \vdots & & & & \ddots \end{array}\right].$$

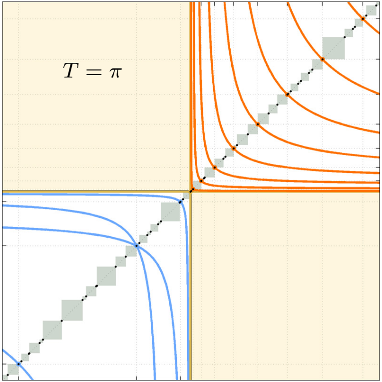
- inadmissible areas
- known branches

# The Fučík Spectrum

$$\Sigma(L) := \{(\alpha, \beta) \in \mathbb{R}^2 : \ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\}$$

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t) & \text{on } (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 & \text{on } \mathbb{R}, \\ u(x,t) = u(x,t+T) & \text{on } (0,\pi) \times \mathbb{R}. \end{cases}$$

$\Sigma(L^{\mathtt{beam}})$




$$\lambda_{k,l} = (2\pi/T)^2 l^2 - k^4, \qquad k \in \mathbb{N}, \ l \in \mathbb{N}_0,$$

$$\Lambda := \left[\begin{array}{c|ccccc} \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \cdots \\ \hline \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \cdots \\ \lambda_{3,0} & \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \\ \lambda_{4,0} & \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \\ \vdots & \vdots & & & & \ddots \end{array}\right].$$

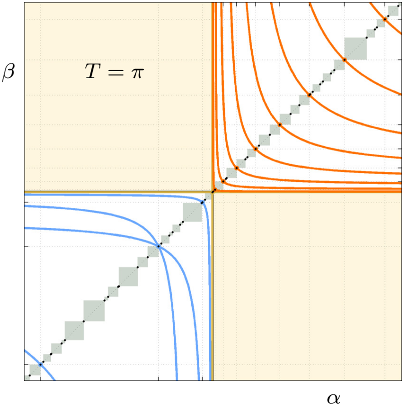
- inadmissible areas
- known branches

# The Fučík Spectrum

$$\Sigma(L) := \left\{(\alpha,\beta)\in\mathbb{R}^2 :\ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\right\}$$

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t) & \text{on } (0,\pi)\times\mathbb{R}, \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 \text{ on } \mathbb{R}, & \\ u(x,t) = u(x,t+T) & \text{on } (0,\pi)\times\mathbb{R}. \end{cases}$$

$\Sigma(L^{\mathtt{beam}})$




$$\lambda_{k,l} = (2\pi/T)^2 l^2 - k^4, \qquad k\in\mathbb{N},\ l\in\mathbb{N}_0,$$

$$\Lambda := \left[\begin{array}{c|ccccc} \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \cdots \\ \hline \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \cdots \\ \lambda_{3,0} & \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \\ \lambda_{4,0} & \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \\ \vdots & \vdots & & & & \ddots \end{array}\right].$$

- inadmissible areas
- known branches
- local & global existence

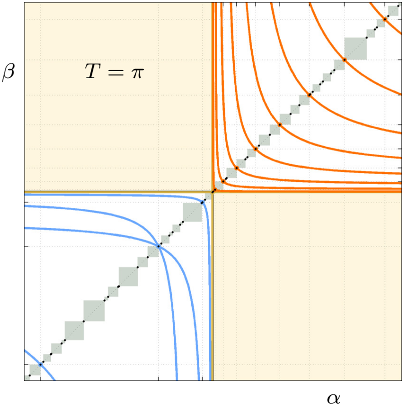
(**Ben-Naoum**, **Fabry** and **Smets**)

# The Fučík Spectrum

$$\Sigma(L) := \{(\alpha,\beta)\in\mathbb{R}^2 :\ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\}$$

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t) & \text{on } (0,\pi)\times\mathbb{R}, \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 & \text{on } \mathbb{R}, \\ u(x,t) = u(x,t+T) & \text{on } (0,\pi)\times\mathbb{R}. \end{cases}$$

$\Sigma(L^{\mathtt{beam}})$



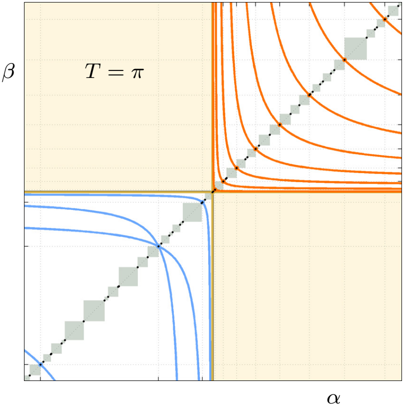

- numerical experiments
  - continuation method
  - shooting method
  - intersections of branches
  - different behaviour

# The Fučík Spectrum

$$\Sigma(L) := \{(\alpha,\beta)\in\mathbb{R}^2 :\ Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\}$$

$$\begin{cases} u_{tt}(x,t) + u_{xxxx}(x,t) + bu^+(x,t) = h(x,t) & \text{on } (0,\pi)\times\mathbb{R}, \\ u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0 & \text{on } \mathbb{R}, \\ u(x,t) = u(x,t+T) & \text{on } (0,\pi)\times\mathbb{R}. \end{cases}$$

$\Sigma(L^{\text{beam}})$




- numerical experiments
  - continuation method
  - shooting method
  - intersections of branches
  - different behaviour
- goals
  - design stable algorithm
  - locate asymptotes of branches

# Variational Approach

$$Lu = \alpha u^{+} - \beta u^{-}$$

# Variational Approach

$$Lu = \alpha u^+ - \beta u^-$$

$$\begin{cases} v = (\mu I - L)u, \\ m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \quad \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}. \end{cases}$$

# Variational Approach

$$Lu = \alpha u^+ - \beta u^-$$

$$(\mu I - L)^{-1} v = \lambda (v + m|v|)$$

$$\begin{cases} v = (\mu I - L)u, \\ m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \quad \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}. \end{cases}$$

# Variational Approach

$$Lu = \alpha u^+ - \beta u^-$$

$$\begin{cases} u = (\mu I - L)^{-1} v, \\ \alpha = \mu - \dfrac{1}{\lambda(1+m)}, \quad \beta = \mu - \dfrac{1}{\lambda(1-m)}. \end{cases}$$

$$(\mu I - L)^{-1} v = \lambda(v + m|v|)$$

$$\begin{cases} v = (\mu I - L)u, \\ m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \quad \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}. \end{cases}$$

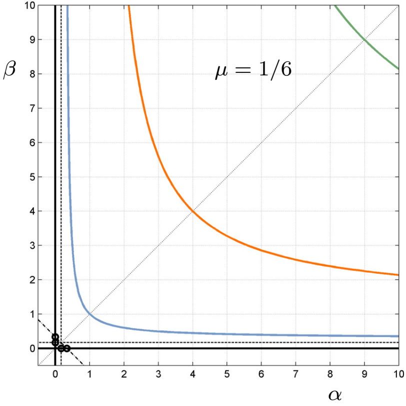
# Variational Approach

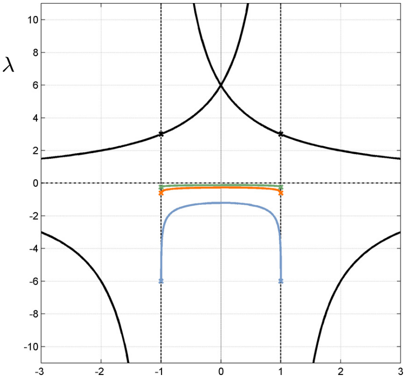
$$Lu = \alpha u^+ - \beta u^-$$

$$\begin{cases} u = (\mu I - L)^{-1} v, \\ \alpha = \mu - \dfrac{1}{\lambda(1+m)}, \quad \beta = \mu - \dfrac{1}{\lambda(1-m)}. \end{cases}$$

$$(\mu I - L)^{-1} v = \lambda(v + m|v|)$$

$$\begin{cases} v = (\mu I - L)u, \\ m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \quad \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}. \end{cases}$$



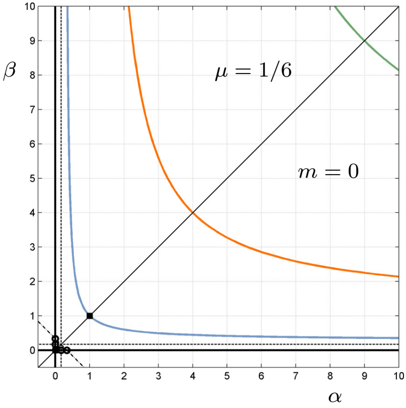
# Variational Approach

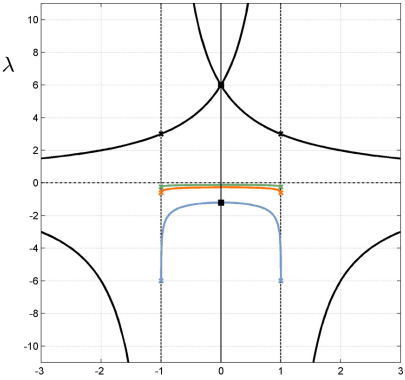
$$Lu = \alpha u^+ - \beta u^-$$

$$\begin{cases} u = (\mu I - L)^{-1} v, \\ \alpha = \mu - \dfrac{1}{\lambda(1+m)}, \quad \beta = \mu - \dfrac{1}{\lambda(1-m)}. \end{cases}$$

$$(\mu I - L)^{-1} v = \lambda(v + m|v|)$$

$$\begin{cases} v = (\mu I - L) u, \\ m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \quad \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}. \end{cases}$$



# Variational Approach

$$Lu = \alpha u^+ - \beta u^-$$

$$\begin{cases} u = (\mu I - L)^{-1} v, \\ \alpha = \mu - \dfrac{1}{\lambda(1+m)}, \quad \beta = \mu - \dfrac{1}{\lambda(1-m)}. \end{cases}$$

$$J(v) = \frac{F(v)}{G(v)},$$

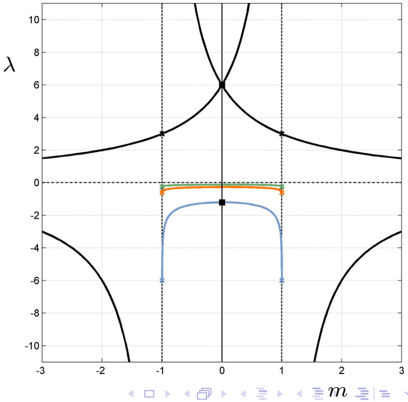
$$F(v) = \frac{1}{2}\int_\Omega (\mu I - L)^{-1} v \cdot v \, \mathrm{d}x,$$

$$G(v) = \frac{1}{2}\int_\Omega v^2 + m|v|v \, \mathrm{d}x,$$

$$\min_v J(v), \quad \max_v J(v).$$

$$(\mu I - L)^{-1} v = \lambda(v + m|v|)$$

$$\begin{cases} v = (\mu I - L)u, \\ m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \quad \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}. \end{cases}$$



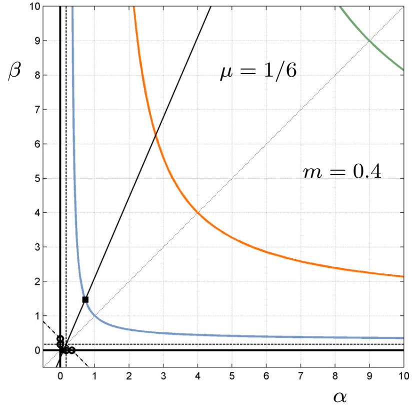
# Variational Approach

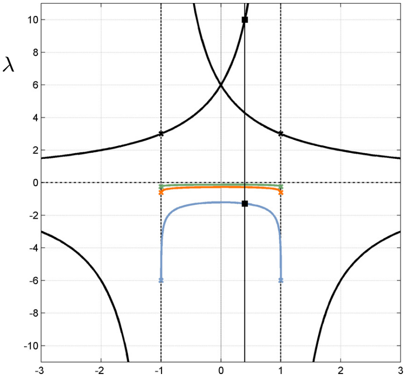
$$Lu = \alpha u^+ - \beta u^-$$

$$\begin{cases} u = (\mu I - L)^{-1} v, \\ \alpha = \mu - \dfrac{1}{\lambda(1+m)}, \quad \beta = \mu - \dfrac{1}{\lambda(1-m)}. \end{cases}$$

$$(\mu I - L)^{-1} v = \lambda(v + m|v|)$$

$$\begin{cases} v = (\mu I - L) u, \\ m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \quad \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}. \end{cases}$$



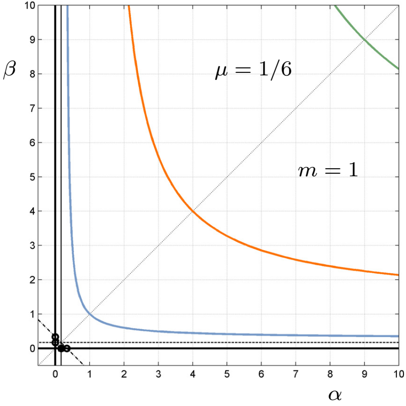
# Variational Approach

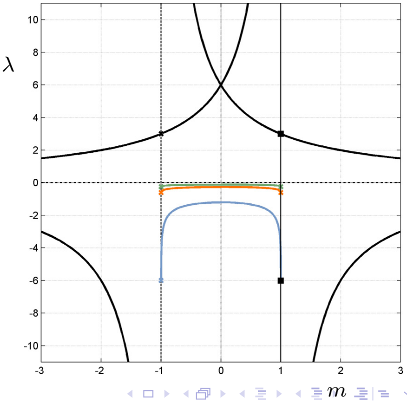
$$Lu = \alpha u^+ - \beta u^-$$

$$\begin{cases} u = (\mu I - L)^{-1} v, \\ \alpha = \mu - \dfrac{1}{\lambda(1+m)}, \quad \beta = \mu - \dfrac{1}{\lambda(1-m)}. \end{cases}$$

$$(\mu I - L)^{-1} v = \lambda(v + m|v|)$$

$$\begin{cases} v = (\mu I - L)u, \\ m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \quad \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}. \end{cases}$$



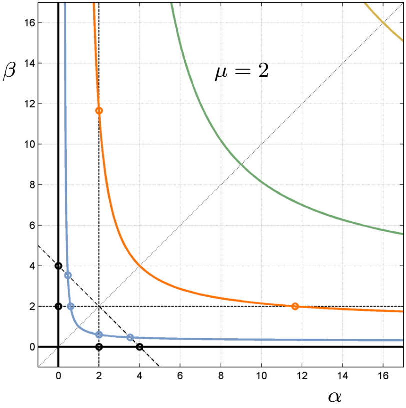
# Variational Approach

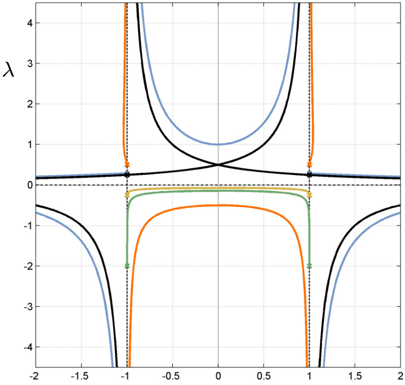
$$Lu = \alpha u^+ - \beta u^-$$

$$\begin{cases} u = (\mu I - L)^{-1} v, \\ \alpha = \mu - \dfrac{1}{\lambda(1+m)}, \quad \beta = \mu - \dfrac{1}{\lambda(1-m)}. \end{cases}$$

$$(\mu I - L)^{-1} v = \lambda(v + m|v|)$$

$$\begin{cases} v = (\mu I - L)u, \\ m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \quad \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}. \end{cases}$$



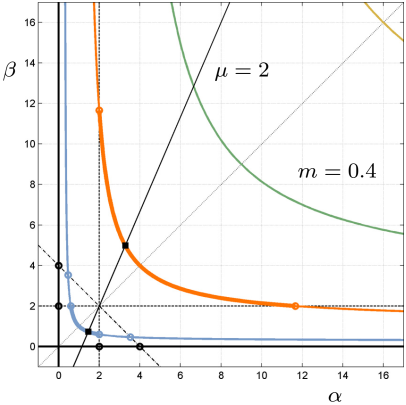
# Variational Approach

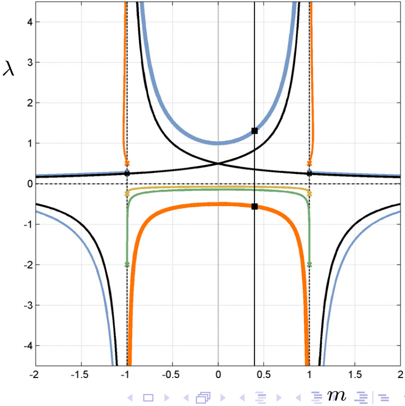
$$Lu = \alpha u^+ - \beta u^-$$

$$\begin{cases} u = (\mu I - L)^{-1} v, \\ \alpha = \mu - \dfrac{1}{\lambda(1+m)}, \quad \beta = \mu - \dfrac{1}{\lambda(1-m)}. \end{cases}$$

$$(\mu I - L)^{-1} v = \lambda(v + m|v|)$$

$$\begin{cases} v = (\mu I - L)u, \\ m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \quad \lambda = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta)}. \end{cases}$$



# Variational Approach - Revision

$Lu = \alpha u^+ - \beta u^-$

# Variational Approach - Revision

$Lu = \alpha u^+ - \beta u^-$

$$\begin{cases} v = (\mu I - L)u, \quad m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \\ \tilde{\lambda} = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta) + \delta(2\mu - \alpha - \beta)}. \end{cases}$$

# Variational Approach - Revision

$$Lu = \alpha u^+ - \beta u^-$$

$$((\mu+\delta)I - L)^{-1}v =$$

$$\tilde{\lambda}(v + m(I - \delta[(\mu+\delta)I - L]^{-1})|v|)$$

$$\begin{cases} v = (\mu I - L)u, \quad m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \\ \tilde{\lambda} = \dfrac{2\mu - \alpha - \beta}{2(\mu-\alpha)(\mu-\beta) + \delta(2\mu - \alpha - \beta)}. \end{cases}$$

# Variational Approach - Revision

$$Lu = \alpha u^+ - \beta u^-$$

$$((\mu+\delta)I - L)^{-1}v = \tilde{\lambda}(v + m(I - \delta[(\mu+\delta)I - L]^{-1})|v|)$$

$$\begin{cases} u = (\mu I - L)^{-1}v, \\ \alpha = \mu - \dfrac{1-\delta\tilde{\lambda}}{\tilde{\lambda}(1+m)}, \quad \beta = \mu - \dfrac{1-\delta\tilde{\lambda}}{\tilde{\lambda}(1-m)}. \end{cases}$$

$$\begin{cases} v = (\mu I - L)u, \quad m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \\ \tilde{\lambda} = \dfrac{2\mu - \alpha - \beta}{2(\mu-\alpha)(\mu-\beta) + \delta(2\mu - \alpha - \beta)}. \end{cases}$$

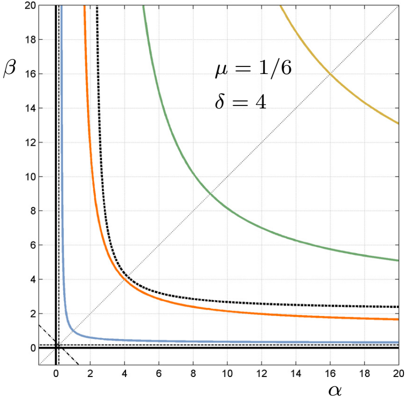
# Variational Approach - Revision

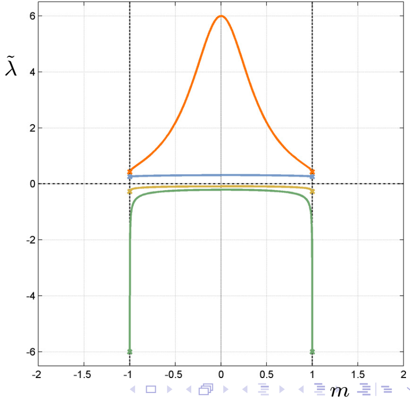
$$Lu = \alpha u^+ - \beta u^-$$

$$((\mu+\delta)I - L)^{-1}v = \tilde\lambda\left(v + m\left(I - \delta[(\mu+\delta)I - L]^{-1}\right)|v|\right)$$

$$\begin{cases} u = (\mu I - L)^{-1} v, \\ \alpha = \mu - \dfrac{1-\delta\tilde\lambda}{\tilde\lambda(1+m)}, \quad \beta = \mu - \dfrac{1-\delta\tilde\lambda}{\tilde\lambda(1-m)}. \end{cases}$$

$$\begin{cases} v = (\mu I - L)u, \quad m = \dfrac{\beta-\alpha}{\beta+\alpha-2\mu}, \\ \tilde\lambda = \dfrac{2\mu-\alpha-\beta}{2(\mu-\alpha)(\mu-\beta)+\delta(2\mu-\alpha-\beta)}. \end{cases}$$



# Variational Approach - Revision

$$Lu = \alpha u^+ - \beta u^-$$

$$((\mu+\delta)I - L)^{-1}v = \tilde\lambda(v + m(I - \delta[(\mu+\delta)I - L]^{-1})|v|)$$

$$\begin{cases} u = (\mu I - L)^{-1}v, \\ \alpha = \mu - \dfrac{1-\delta\tilde\lambda}{\tilde\lambda(1+m)}, \quad \beta = \mu - \dfrac{1-\delta\tilde\lambda}{\tilde\lambda(1-m)}. \end{cases}$$

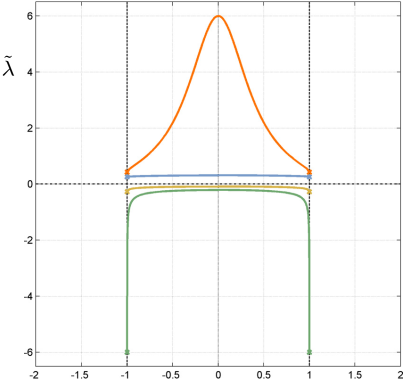
$$\begin{cases} v = (\mu I - L)u, \quad m = \dfrac{\beta-\alpha}{\beta+\alpha-2\mu}, \\ \tilde\lambda = \dfrac{2\mu-\alpha-\beta}{2(\mu-\alpha)(\mu-\beta)+\delta(2\mu-\alpha-\beta)}. \end{cases}$$

$$J(v) = \frac{F(v)}{G(v)},$$

$$F(v) = \frac{1}{2}\int_\Omega ((\mu+\delta)I - L)^{-1}v \cdot v \,\mathrm{d}x,$$

$$G(v) = \frac{1}{2}\int_\Omega v^2 + m(I - \delta[(\mu+\delta)I - L]^{-1})|v|v \,\mathrm{d}x,$$

$$\min_v J(v), \quad \max_v J(v).$$



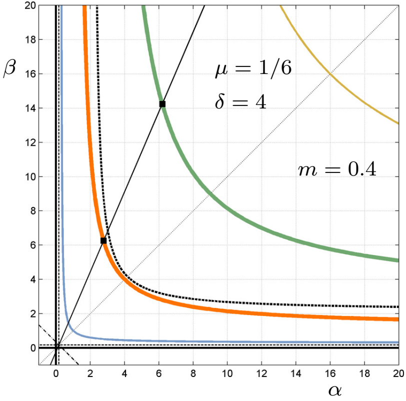
# Variational Approach - Revision

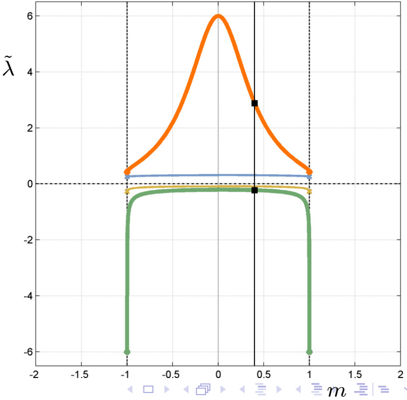
$$Lu = \alpha u^+ - \beta u^-$$

$$((\mu+\delta)I - L)^{-1}v = \tilde{\lambda}(v + m(I - \delta[(\mu+\delta)I - L]^{-1})|v|)$$

$$\begin{cases} u = (\mu I - L)^{-1}v, \\ \alpha = \mu - \dfrac{1-\delta\tilde{\lambda}}{\tilde{\lambda}(1+m)}, \quad \beta = \mu - \dfrac{1-\delta\tilde{\lambda}}{\tilde{\lambda}(1-m)}. \end{cases}$$

$$\begin{cases} v = (\mu I - L)u, \quad m = \dfrac{\beta-\alpha}{\beta+\alpha-2\mu}, \\ \tilde{\lambda} = \dfrac{2\mu-\alpha-\beta}{2(\mu-\alpha)(\mu-\beta)+\delta(2\mu-\alpha-\beta)}. \end{cases}$$



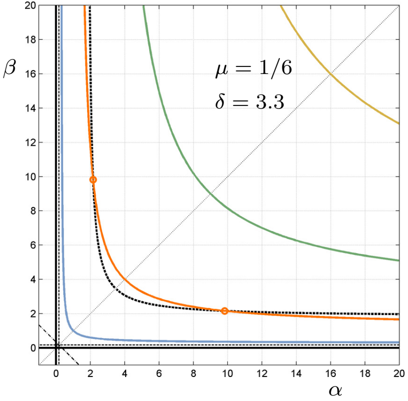
# Variational Approach - Revision

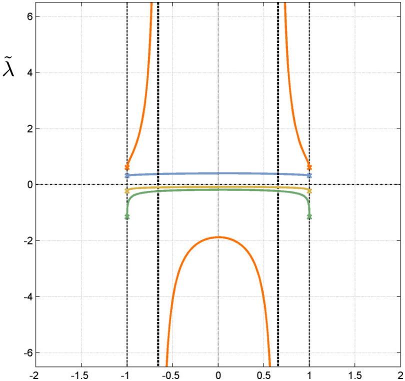
$$Lu = \alpha u^+ - \beta u^-$$

$$((\mu+\delta)I - L)^{-1}v = \tilde\lambda(v + m(I - \delta[(\mu+\delta)I - L]^{-1})|v|)$$

$$\begin{cases} u = (\mu I - L)^{-1}v, \\ \alpha = \mu - \dfrac{1-\delta\tilde\lambda}{\tilde\lambda(1+m)}, \quad \beta = \mu - \dfrac{1-\delta\tilde\lambda}{\tilde\lambda(1-m)}. \end{cases}$$

$$\begin{cases} v = (\mu I - L)u, \quad m = \dfrac{\beta-\alpha}{\beta+\alpha-2\mu}, \\ \tilde\lambda = \dfrac{2\mu-\alpha-\beta}{2(\mu-\alpha)(\mu-\beta)+\delta(2\mu-\alpha-\beta)}. \end{cases}$$



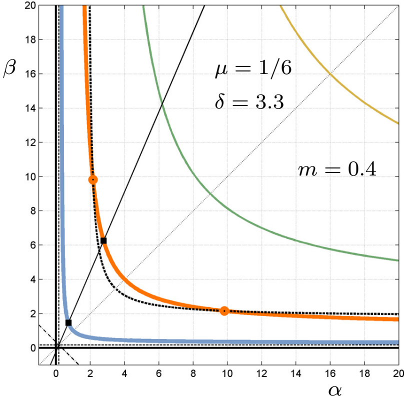
# Variational Approach - Revision

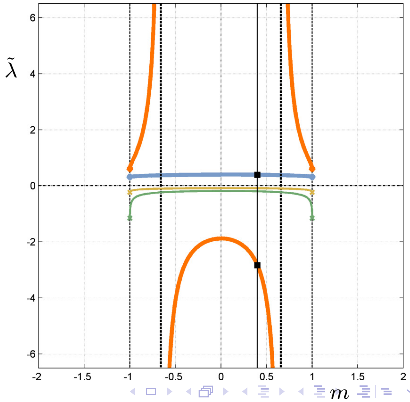
$$Lu = \alpha u^+ - \beta u^-$$

$$((\mu+\delta)I - L)^{-1}v = \tilde{\lambda}(v + m(I - \delta[(\mu+\delta)I - L]^{-1})|v|)$$

$$\begin{cases} u = (\mu I - L)^{-1}v, \\ \alpha = \mu - \dfrac{1-\delta\tilde{\lambda}}{\tilde{\lambda}(1+m)}, \quad \beta = \mu - \dfrac{1-\delta\tilde{\lambda}}{\tilde{\lambda}(1-m)}. \end{cases}$$

$$\begin{cases} v = (\mu I - L)u, \quad m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \\ \tilde{\lambda} = \dfrac{2\mu - \alpha - \beta}{2(\mu-\alpha)(\mu-\beta)+\delta(2\mu - \alpha - \beta)}. \end{cases}$$



# Example - Neumann Wave Operator

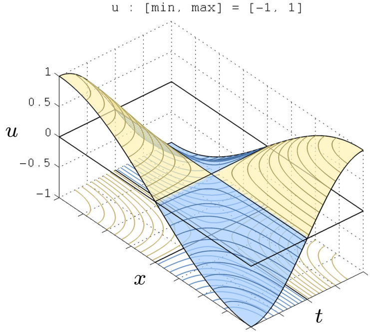
$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha u^{+}(x,t) - \beta u^{-}(x,t) = 0, & (x,t) \in (0,X) \times (0,T), \\ u_t\,(x,0) = u_t\,(x,T) = 0, & x \quad \in [0,X], \\ u_x\,(0,t) = u_x\,(X,t) = 0, & t \quad \in [0,T]. \end{cases}$$

# Example - Neumann Wave Operator

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha u^+(x,t) - \beta u^-(x,t) = 0, & (x,t) \in (0,X) \times (0,T), \\ u_t(x,0) = u_t(x,T) = 0, & x \in [0,X], \\ u_x(0,t) = u_x(X,t) = 0, & t \in [0,T]. \end{cases}$$

- $X = 2, \quad T = 1,$

$$\lambda_{k,l} = (\pi/T)^2 l^2 - (\pi/X)^2 k^2, \qquad k, l \in \mathbb{N}_0,$$



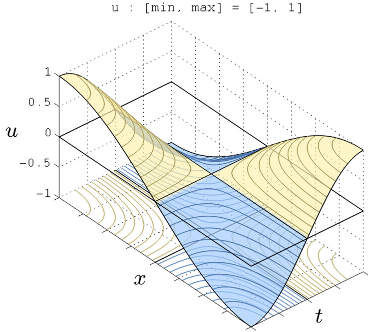

$$\Lambda \doteq \left[\begin{array}{r|rrrr} 0 & 9.870 & 39.478 & 88.826 & \dots \\ \hline -2.467 & 7.402 & 37.011 & 86.359 & \dots \\ -9.870 & 0 & 29.609 & 78.957 & \\ -22.207 & -12.337 & 17.272 & 66.620 & \\ -39.478 & -29.609 & 0 & 49.348 & \\ \vdots & \vdots & & & \ddots \end{array}\right].$$

# Example - Neumann Wave Operator

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha u^+(x,t) - \beta u^-(x,t) = 0, & (x,t) \in (0,X) \times (0,T), \\ u_t\,(x,0) = u_t\,(x,T) = 0, & x \quad \in [0,X], \\ u_x\,(0,t) = u_x\,(X,t) = 0, & t \quad \in [0,T]. \end{cases}$$

- $X = 2, \quad T = 1,$
- $\mu = 1, \quad \delta = 0, \quad m = 1,$

$$\lambda_{k,l} = (\pi/T)^2 l^2 - (\pi/X)^2 k^2, \qquad k, l \in \mathbb{N}_0,$$

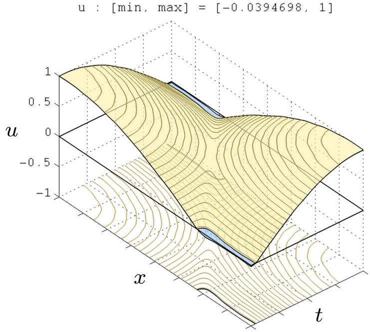



$$\Lambda \doteq \left[\begin{array}{c|cccc} 0 & 9.870 & 39.478 & 88.826 & \dots \\ \hline -2.467 & 7.402 & 37.011 & 86.359 & \dots \\ -9.870 & 0 & 29.609 & 78.957 & \\ -22.207 & -12.337 & 17.272 & 66.620 & \\ -39.478 & -29.609 & 0 & 49.348 & \\ \vdots & \vdots & & & \ddots \end{array}\right].$$

# Example - Neumann Wave Operator

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha u^+(x,t) - \beta u^-(x,t) = 0, & (x,t) \in (0,X) \times (0,T), \\ u_t\,(x,0) = u_t\,(x,T) = 0, & x \in [0,X], \\ u_x\,(0,t) = u_x\,(X,t) = 0, & t \in [0,T]. \end{cases}$$

- $X = 2, \quad T = 1,$
- $\mu = 1, \quad \delta = 0, \quad m = 1,$
- $\alpha = 1.587, \quad \beta = +\infty,$
  $\tilde{\lambda} = 0.851, \quad \|J'(v)\|_2 = 2.3 \cdot 10^{-5}.$



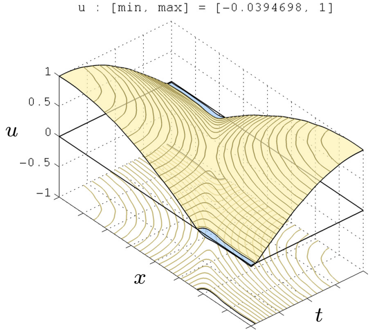

$$\lambda_{k,l} = (\pi/T)^2 l^2 - (\pi/X)^2 k^2, \qquad k,l \in \mathbb{N}_0,$$

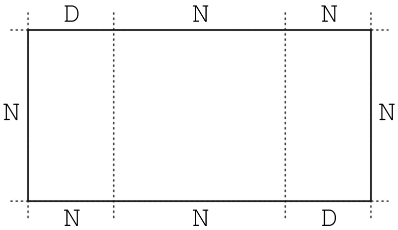
$$\Lambda \doteq \left[\begin{array}{r|rrrc} 0 & 9.870 & 39.478 & 88.826 & \dots \\ \hline -2.467 & 7.402 & 37.011 & 86.359 & \dots \\ -9.870 & 0 & 29.609 & 78.957 & \\ -22.207 & -12.337 & 17.272 & 66.620 & \\ -39.478 & -29.609 & 0 & 49.348 & \\ \vdots & \vdots & & & \ddots \end{array}\right].$$

# Example - Neumann Wave Operator

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha u^+(x,t) - \beta u^-(x,t) = 0, & (x,t) \in (0,X) \times (0,T), \\ u_t(x,0) = u_t(x,T) = 0, & x \in [0,X], \\ u_x(0,t) = u_x(X,t) = 0, & t \in [0,T]. \end{cases}$$

- $X = 2, \quad T = 1,$
- $\mu = 1, \quad \delta = 0, \quad m = 1,$
- $\alpha = 1.587, \quad \beta = +\infty,$
  $\tilde{\lambda} = 0.851, \quad \|J'(v)\|_2 = 2.3 \cdot 10^{-5}.$

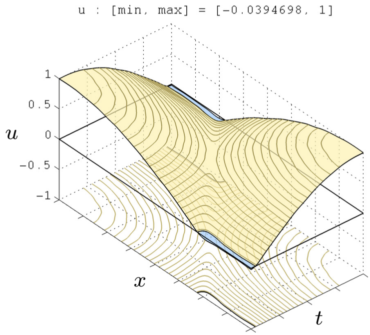


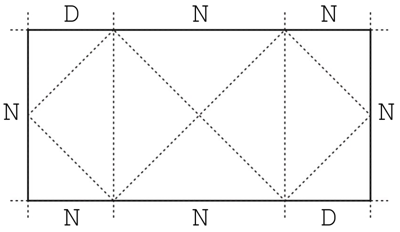



# Example - Neumann Wave Operator

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha u^{+}(x,t) - \beta u^{-}(x,t) = 0, & (x,t) \in (0,X) \times (0,T), \\ u_t\,(x,0) = u_t\,(x,T) = 0, & x \quad \in [0,X], \\ u_x\,(0,t) = u_x\,(X,t) = 0, & t \quad \in [0,T]. \end{cases}$$

- $X = 2, \quad T = 1,$
- $\mu = 1, \quad \delta = 0, \quad m = 1,$
- $\alpha = 1.587, \quad \beta = +\infty,$
  $\tilde{\lambda} = 0.851, \quad \|J'(v)\|_2 = 2.3 \cdot 10^{-5}.$

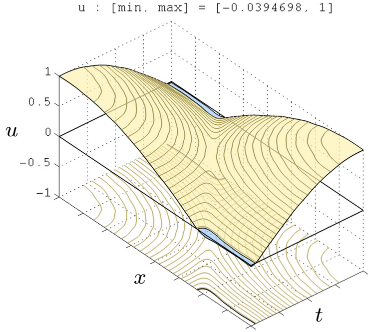


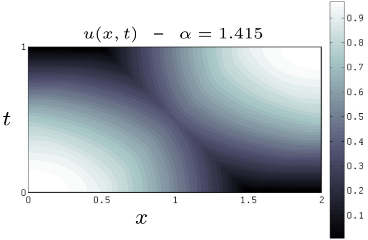



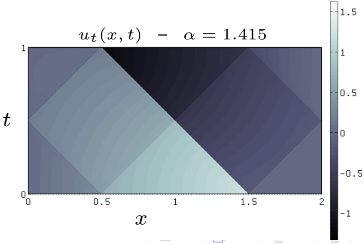
# Example - Neumann Wave Operator

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha u^+(x,t) - \beta u^-(x,t) = 0, & (x,t) \in (0,X) \times (0,T), \\ u_t\,(x,0) = u_t\,(x,T) = 0, & x \in [0,X], \\ u_x\,(0,t) = u_x\,(X,t) = 0, & t \in [0,T]. \end{cases}$$

- $X = 2, \quad T = 1,$
- $\mu = 1, \quad \delta = 0, \quad m = 1,$
- $\alpha = 1.587, \quad \beta = +\infty,$
  $\tilde{\lambda} = 0.851, \quad \|J'(v)\|_2 = 2.3 \cdot 10^{-5}.$



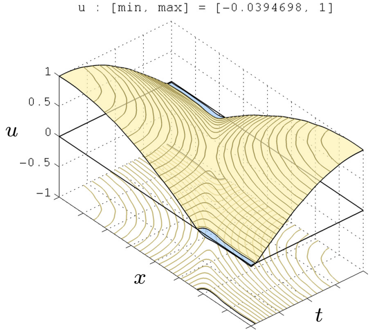



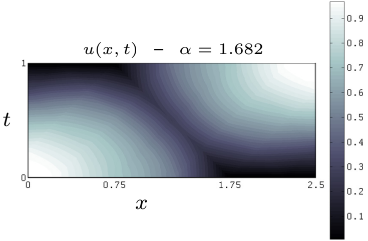



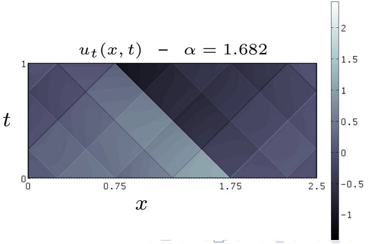
# Example - Neumann Wave Operator

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha u^+(x,t) - \beta u^-(x,t) = 0, & (x,t) \in (0,X) \times (0,T), \\ u_t\,(x,0) = u_t\,(x,T) = 0, & x \quad \in [0,X], \\ u_x\,(0,t) = u_x\,(X,t) = 0, & t \quad \in [0,T]. \end{cases}$$

- $X = 2, \quad T = 1,$
- $\mu = 1, \quad \delta = 0, \quad m = 1,$
- $\alpha = 1.587, \quad \beta = +\infty,$
  $\tilde{\lambda} = 0.851, \quad \|J'(v)\|_2 = 2.3 \cdot 10^{-5}.$



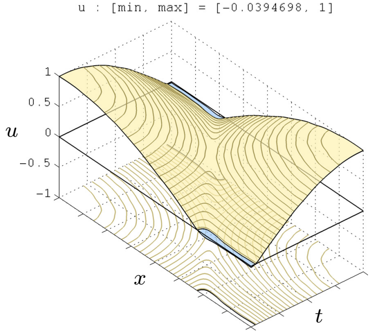



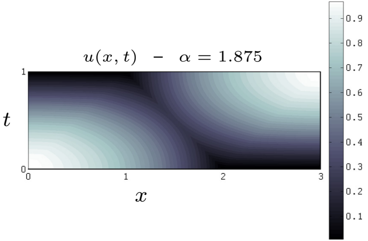



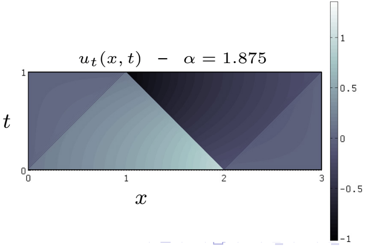
# Example - Neumann Wave Operator

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha u^+(x,t) - \beta u^-(x,t) = 0, & (x,t) \in (0,X) \times (0,T), \\ u_t(x,0) = u_t(x,T) = 0, & x \in [0,X], \\ u_x(0,t) = u_x(X,t) = 0, & t \in [0,T]. \end{cases}$$

- $X = 2, \quad T = 1,$
- $\mu = 1, \quad \delta = 0, \quad m = 1,$
- $\alpha = 1.587, \quad \beta = +\infty,$
  $\tilde{\lambda} = 0.851, \quad \|J'(v)\|_2 = 2.3 \cdot 10^{-5}.$



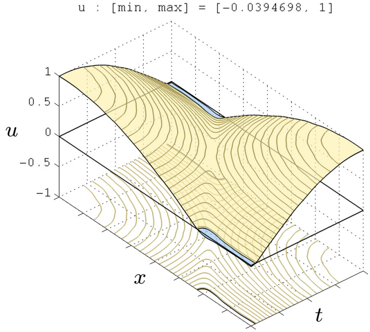



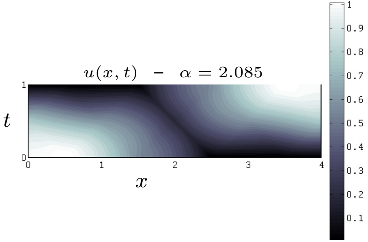



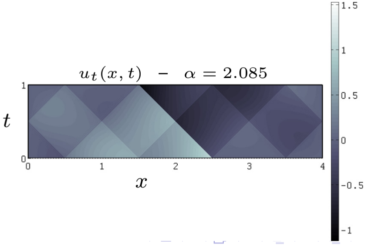
# Example - Neumann Wave Operator

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha u^{+}(x,t) - \beta u^{-}(x,t) = 0, & (x,t) \in (0,X)\times(0,T), \\ u_t\,(x,0) = u_t\,(x,T) = 0, & x \quad \in [0,X], \\ u_x\,(0,t) = u_x\,(X,t) = 0, & t \quad \in [0,T]. \end{cases}$$

- $X = 2, \quad T = 1,$
- $\mu = 1, \quad \delta = 0, \quad m = 1,$
- $\alpha = 1.587, \quad \beta = +\infty,$
  $\tilde{\lambda} = 0.851, \quad \|J'(v)\|_2 = 2.3 \cdot 10^{-5}.$



# Summary

- Variational approach gives us robust and global algorithm.
- Asymptotes of the Fučík curves can be investigated.

# Summary

- Variational approach gives us robust and global algorithm.
- Asymptotes of the Fučík curves can be investigated.

- Outlook
  - Prepare a massive numerical experiment for the wave and the beam operator ($\mu, \delta \in \mathbb{R}$, $m \in (0, 1]$).
  - Clarify the validity of the variational approach.
  - Prove observed qualitative properties of generalized eigenfunctions.

# References


S. Fučík
*Solvability of Nonlinear Equations and Boundary Value Problems*,
D. Reidel Publ. Company, Holland, 1980.

K. Ben-Naoum, C. Fabry and D. Smets
*Structure of the Fučík spectrum and existence of solutions for equations with asymmetric nonlinearities*,
Proc. Roy. Soc. Edinburgh 131 A (2001), 241–265.

J. Horák and W. Reichel
*Analytical and numerical results for the Fučík spectrum of the Laplacian*,
J. Comput. Appl. Math. 161 (2003), 313–338.

L. D. Humphreys
*Numerical mountain pass solutions of a suspension bridge equation*,
Nonlinear Anal. 28 (1997), 1811–1826.

P. Krejčí
*On solvability of equations of the 4th order with jumping nonlinearities*,
Časopis pro pěst. mat. 108 (1983), 29–39.