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The Fučík Spectrum for Multi-Point Boundary Value Problems

Petr Nečesal

Department of Mathematics
Faculty of Applied Sciences, University of West Bohemia in Pilsen

Variational and Topological Methods:
Theory, Applications, Numerical Simulations, and Open Problems

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Outline

The Fučík Spectrum

$\Sigma(L) := \big\{(\alpha, \beta) \in \mathbb{R}^2 : \; Lu = \alpha u^+ - \beta u^-$ has a nontrivial solution$\big\}$

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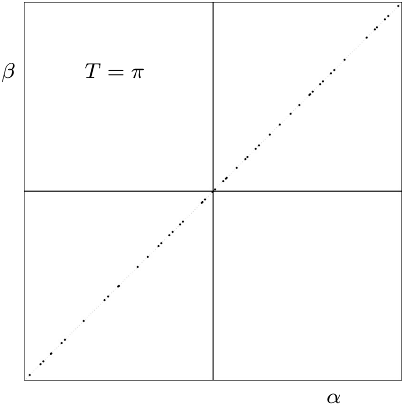

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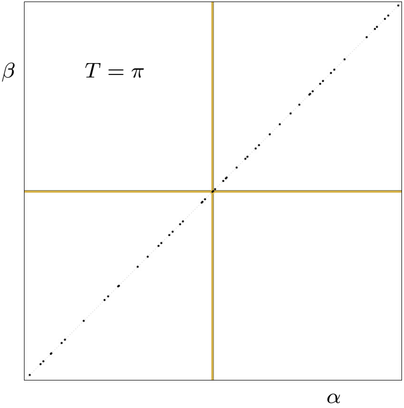

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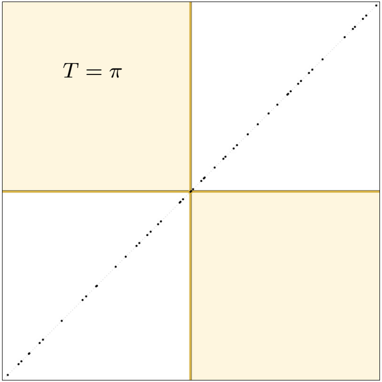

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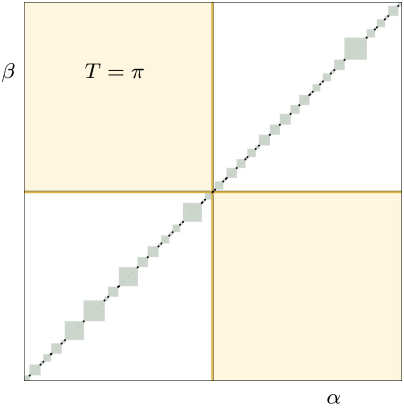

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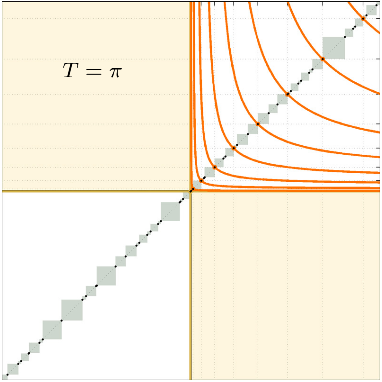

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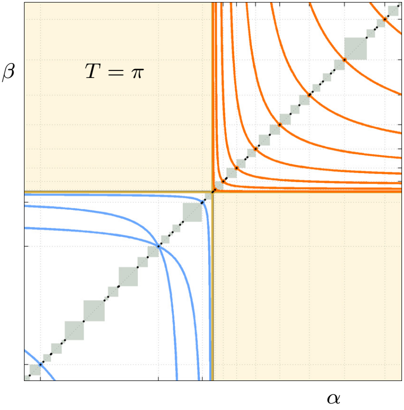

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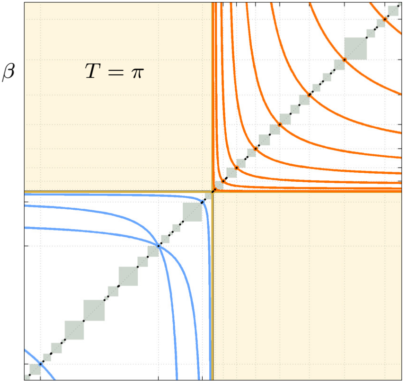

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(**Ben-Naoum**, **Fabry** and **Smets**)

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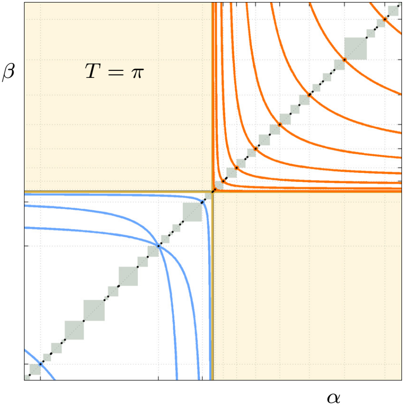

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  - shooting method
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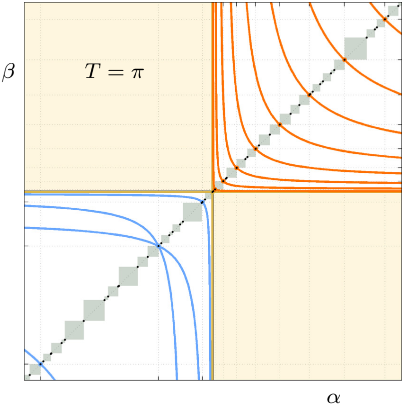

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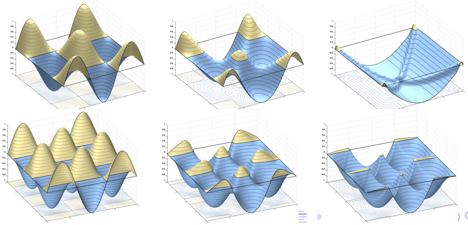

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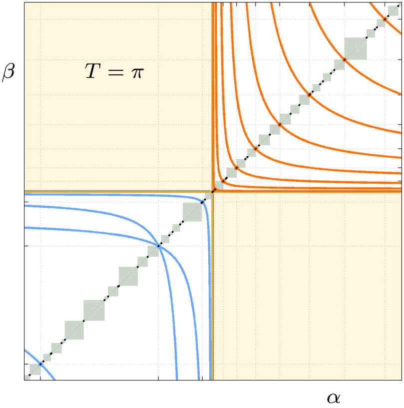

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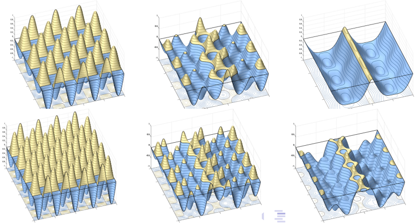

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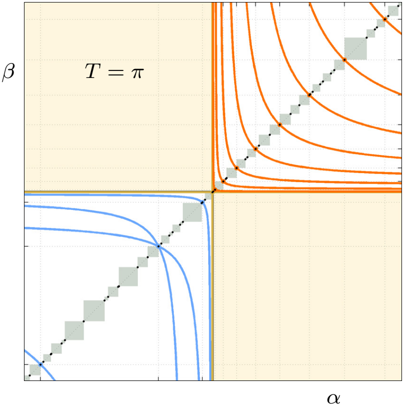

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  - different behaviour
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  - design stable algorithm
  - locate asymptotes of branches

Outline

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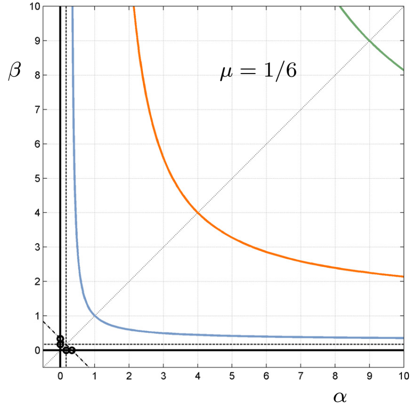

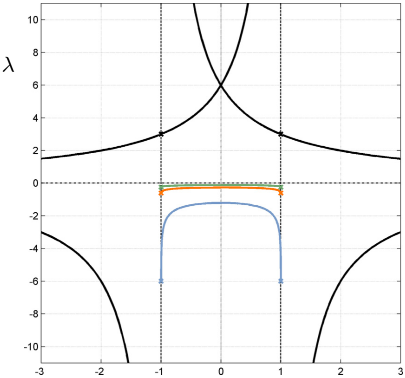

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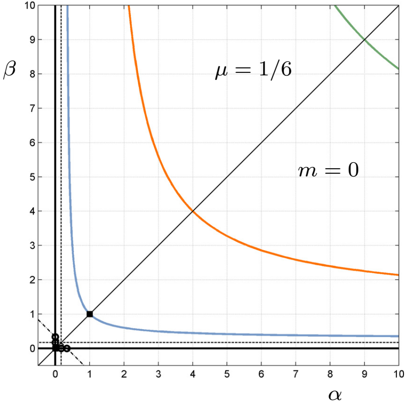

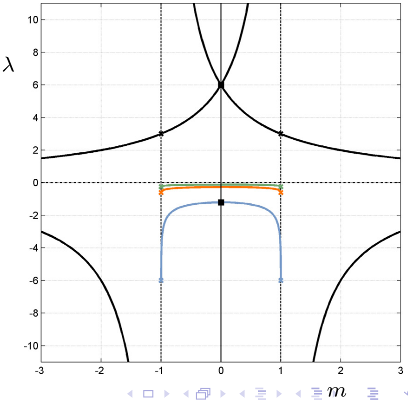

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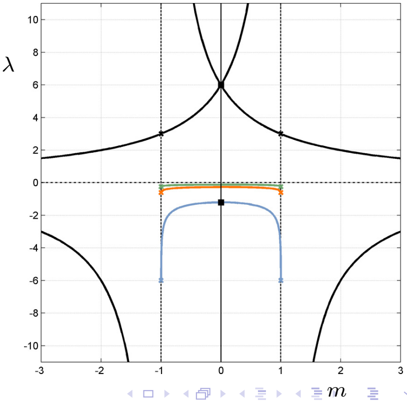
$$F(v) = \frac{1}{2} \int_\Omega (\mu I - L)^{-1} v \cdot v \, \mathrm{d}x,$$

$$G(v) = \frac{1}{2} \int_\Omega v^2 + m|v| v \, \mathrm{d}x,$$

$$\min_v J(v), \quad \max_v J(v).$$



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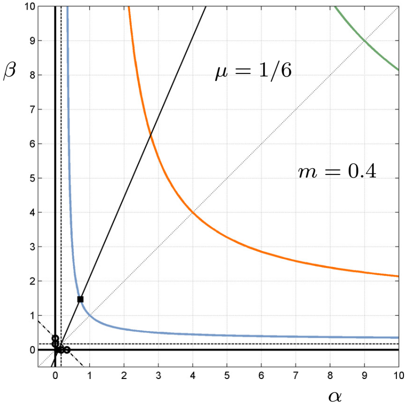

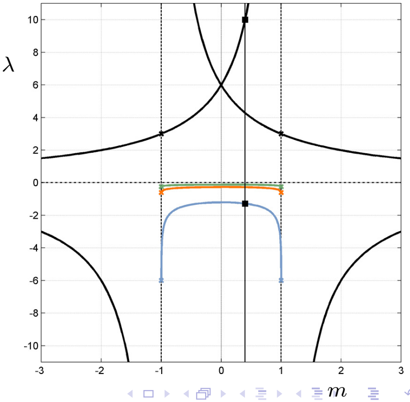

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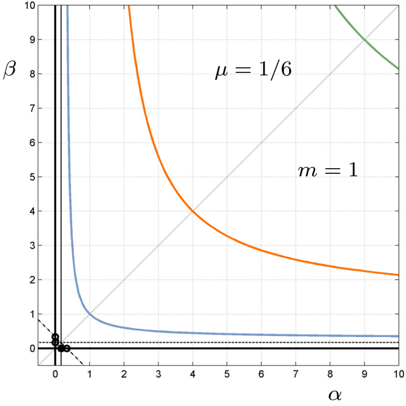

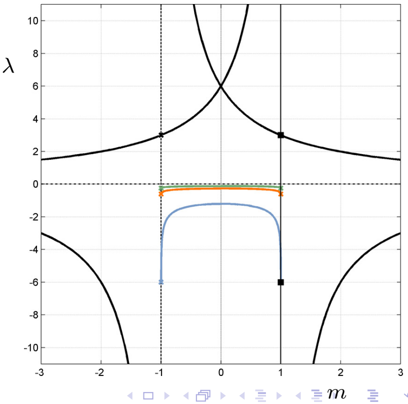

Variational Approach

$$Lu = \alpha u^+ - \beta u^-$$

$$\begin{cases} u = (\mu I - L)^{-1} v, \\ \alpha = \mu - \dfrac{1}{\lambda(1+m)}, \quad \beta = \mu - \dfrac{1}{\lambda(1-m)}. \end{cases}$$

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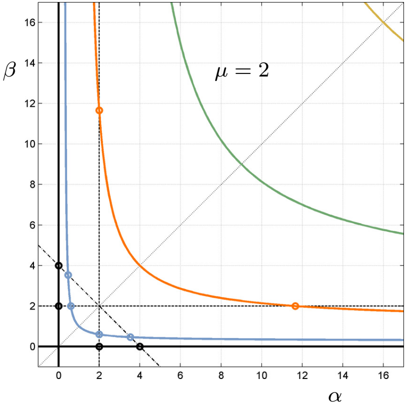

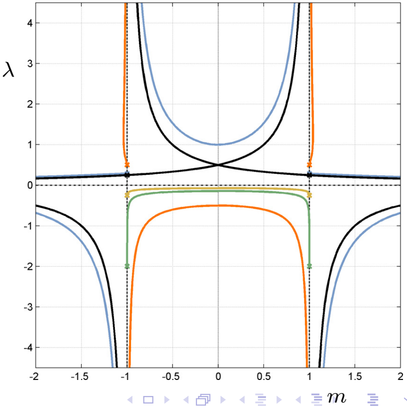

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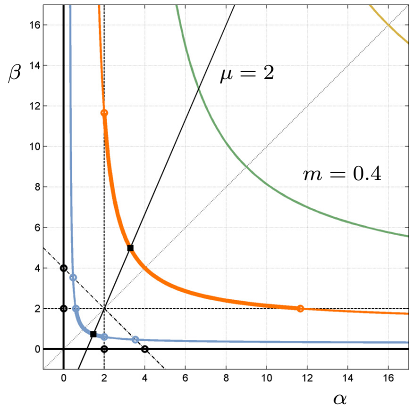

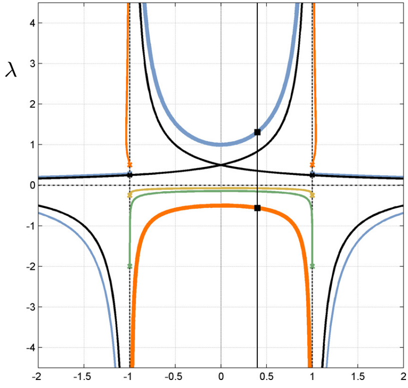

Outline

Variational Approach - Revision

$Lu = \alpha u^+ - \beta u^-$

Variational Approach - Revision

$Lu = \alpha u^+ - \beta u^-$

$$\begin{cases} v = (\mu I - L)u, \quad m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \\ \tilde{\lambda} = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta) + \delta(2\mu - \alpha - \beta)}. \end{cases}$$

Variational Approach - Revision

$$Lu = \alpha u^+ - \beta u^-$$

$$((\mu+\delta)I - L)^{-1}v =$$

$$\tilde{\lambda}(v + m(I - \delta[(\mu+\delta)I - L]^{-1})|v|)$$

$$\begin{cases} v = (\mu I - L)u, \quad m = \dfrac{\beta - \alpha}{\beta + \alpha - 2\mu}, \\ \tilde{\lambda} = \dfrac{2\mu - \alpha - \beta}{2(\mu - \alpha)(\mu - \beta) + \delta(2\mu - \alpha - \beta)}. \end{cases}$$

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Variational Approach - Revision

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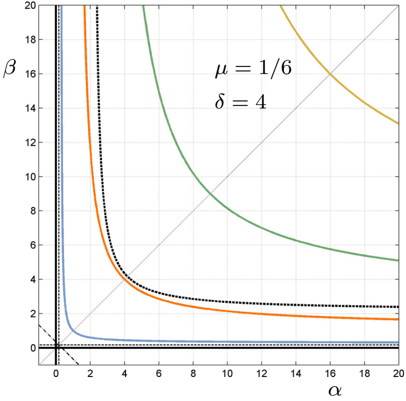

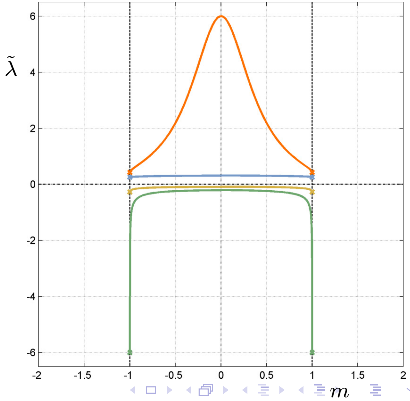

Variational Approach - Revision

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$$\left((\mu+\delta)I - L\right)^{-1} v = \tilde\lambda\left(v + m\left(I - \delta[(\mu+\delta)I - L]^{-1}\right)|v|\right)$$

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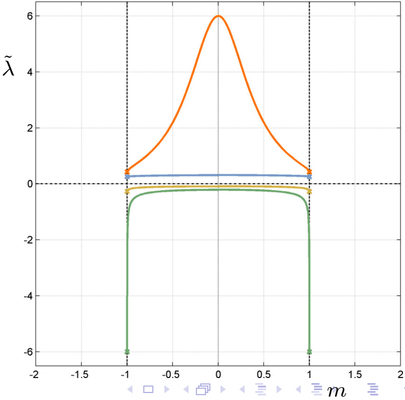
$$J(v) = \frac{F(v)}{G(v)},$$

$$F(v) = \frac{1}{2}\int_\Omega \left((\mu+\delta)I - L\right)^{-1} v \cdot v \,\mathrm{d}x,$$

$$G(v) = \frac{1}{2}\int_\Omega v^2 + m\left(I - \delta[(\mu+\delta)I - L]^{-1}\right)|v| v \,\mathrm{d}x,$$

$$\min_v J(v), \quad \max_v J(v).$$



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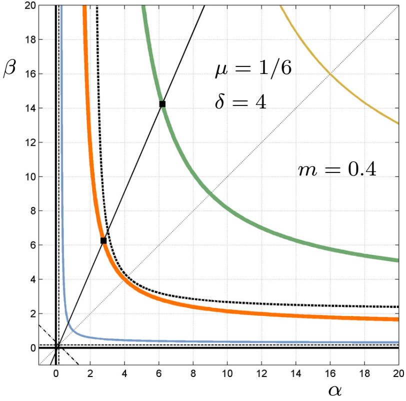

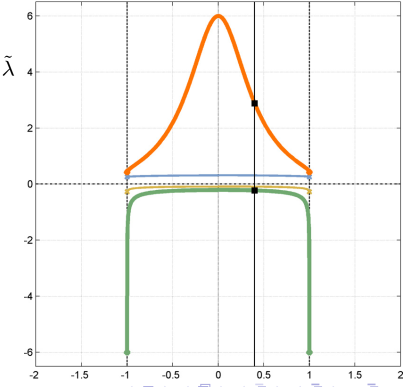

Variational Approach - Revision

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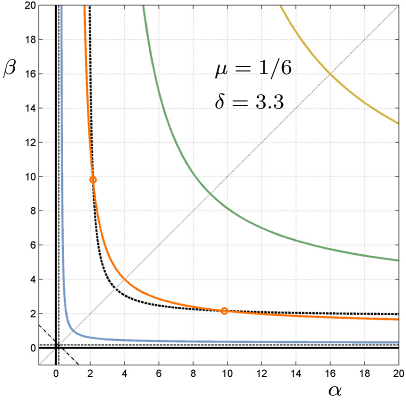

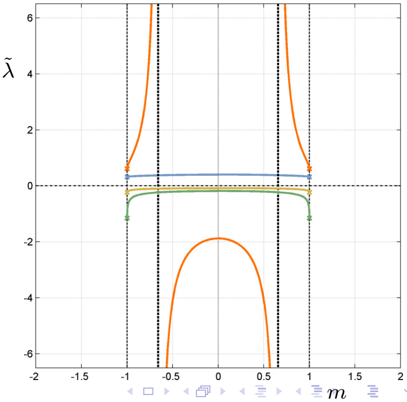

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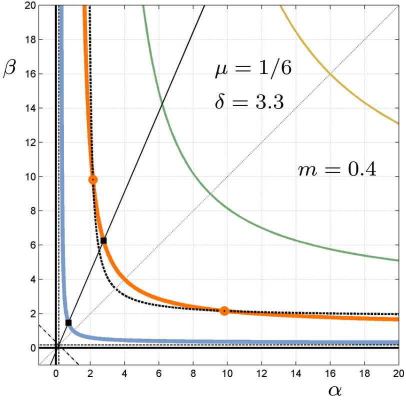
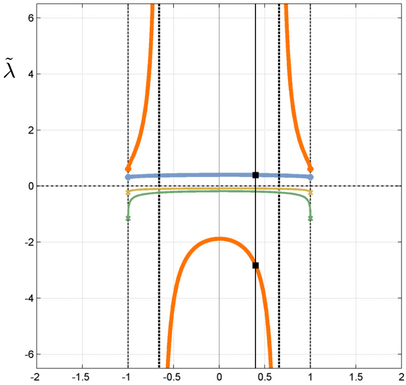
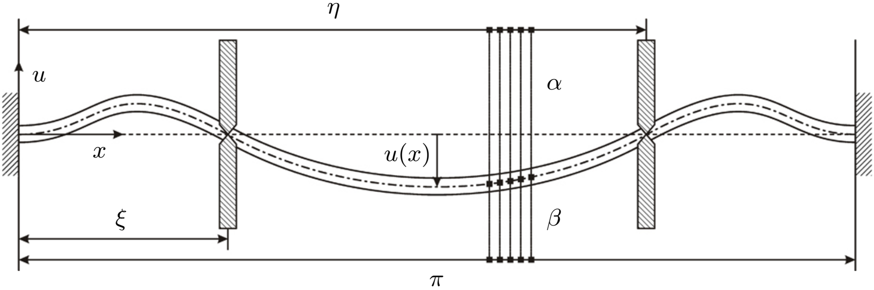




Outline

Multi-Point Selfadjoint Operator

$$\begin{cases} u^{\mathrm{IV}}(x) = \alpha u^+(x) - \beta u^-(x), & x \in (0,\xi) \cup (\xi,\eta) \cup (\eta,\pi), \\ u'(0) = u(0) = u(\xi) = u(\eta) = u(\pi) = u'(\pi) = 0, & 0 < \xi < \eta < \pi. \end{cases} \tag{1}$$



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Definition

Let us define a weak solution as

$$\begin{cases} u \in V, \\ (u'', v'') = (\alpha u^+(x) - \beta u^-(x), v) \qquad \forall\, v \in V, \end{cases} \tag{2}$$

where

$$\begin{aligned} (u,v) &:= \int_0^\pi u(x)v(x)\,\mathrm{d}x, \\ V &:= \left\{ v \in W_0^{2,2}(0,\pi) : v(\xi) = v(\eta) = 0 \right\}, \qquad 0 < \xi < \eta < \pi. \end{aligned}$$

Lemma

If u is a weak solution of (2) then u is the classical solution of (1). Moreover,

$$u \in C^2([0,\pi]) \qquad \textit{and} \qquad u|_{(a,b)} \in C^4(a,b) \quad \textit{for } (a,b) = \begin{cases} (0,\xi), \\ (\xi,\eta), \\ (\eta,\pi). \end{cases}$$

Multi-Point Selfadjoint Operator

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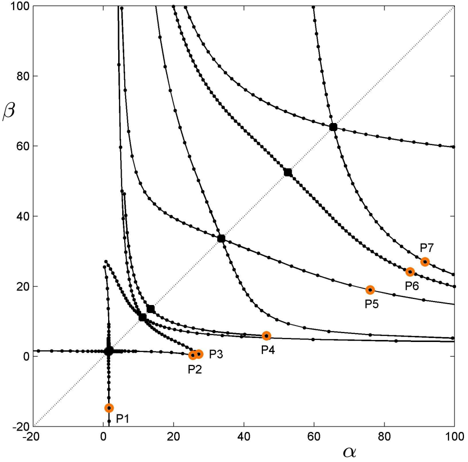
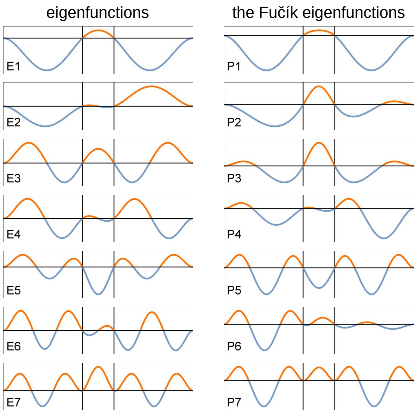
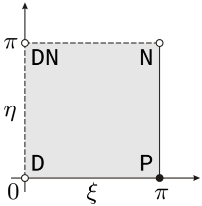




Outline

4-Point Non-Selfadjoint Operator

$$\begin{cases} -u''(x) = \alpha u^+(x) - \beta u^-(x), & x \in [0,\pi], \\ u'(0) = u'(\xi), \quad u(\eta) = u(\pi), & \xi \in (0,\pi], \quad \eta \in [0,\pi). \end{cases} \tag{3}$$



4-Point Non-Selfadjoint Operator

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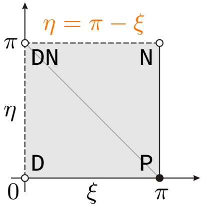

4-Point Non-Selfadjoint Operator

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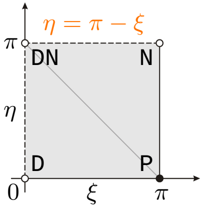

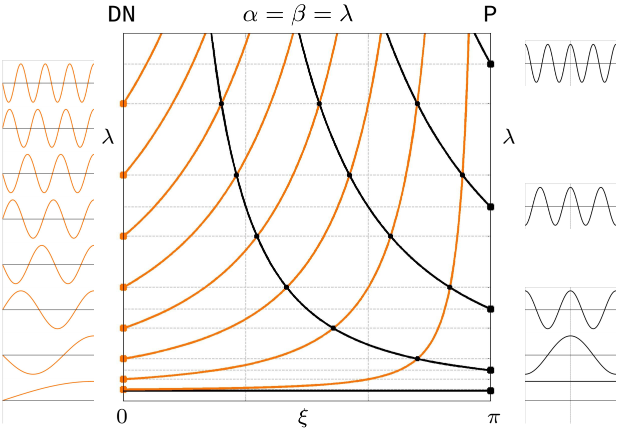

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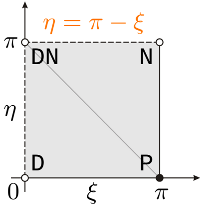

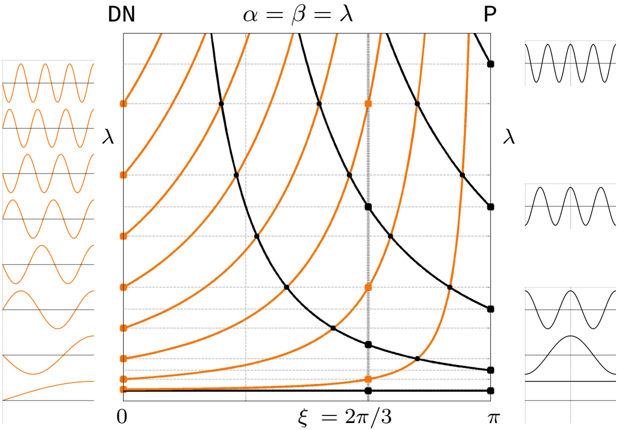

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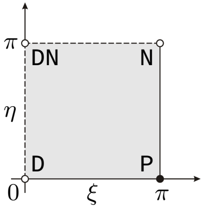
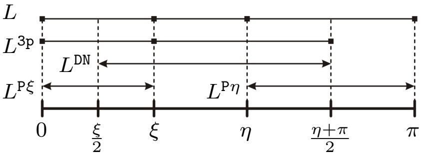




Definition

Let us define the operators

$$Lu := L^{\mathrm{P}\xi}u := L^{\mathrm{P}\eta} := L^{\mathrm{DN}}u := L^{3\mathrm{p}}u := -u''$$

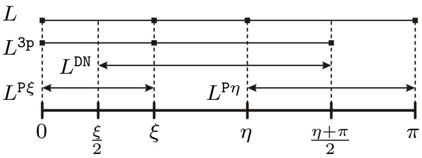
for $\xi \in (0,\pi]$ and $\eta \in [0,\pi)$ by

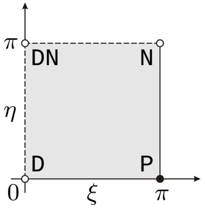
$$D(L) := \{u \in C^2([0,\pi]) : \quad u'(0) = u'(\xi), \quad u(\eta) = u(\pi) \quad \},$$

$$D\left(L^{\mathrm{P}\xi}\right) := \{u \in C^2([0,\pi]) : \quad u'(0) = u'(\xi), \quad u(0) = u(\xi) \quad \},$$
$$D\left(L^{\mathrm{P}\eta}\right) := \{u \in C^2([0,\pi]) : \quad u'(\eta) = u'(\pi), \quad u(\eta) = u(\pi) \quad \},$$

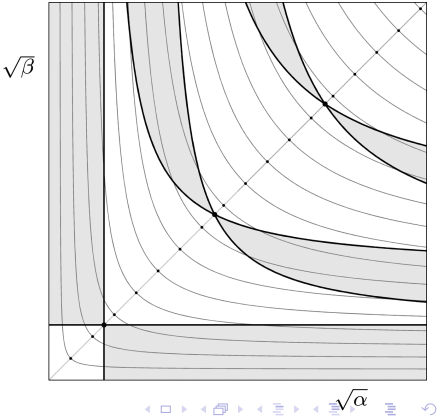
$$D\left(L^{\mathrm{DN}}\right) := \{u \in C^2([0,\pi]) : \quad u\left(\tfrac{\xi}{2}\right) = 0, \quad u'\left(\tfrac{\eta+\pi}{2}\right) = 0 \quad \},$$
$$D\left(L^{3\mathrm{p}}\right) := \{u \in C^2([0,\pi]) : \quad u'(0) = u'(\xi), \quad u'\left(\tfrac{\eta+\pi}{2}\right) = 0, \quad u(0)u(\xi) \le 0\}.$$

4-Point Non-Selfadjoint Operator

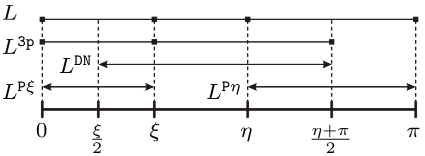


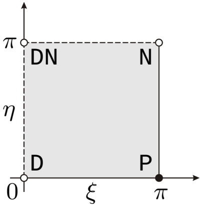


- $\sigma(L) = \sigma(L^{P\xi}) \cup \sigma(L^{P\eta}) \cup \sigma(L^{DN})$,



4-Point Non-Selfadjoint Operator





- $\sigma(L) = \sigma(L^{P\xi}) \cup \sigma(L^{P\eta}) \cup \sigma(L^{DN})$,
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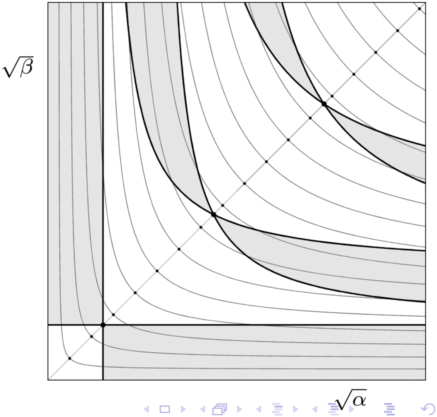
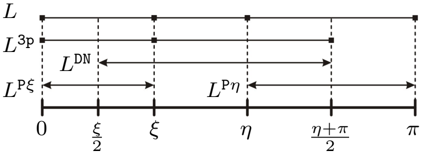
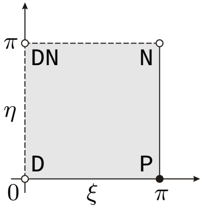
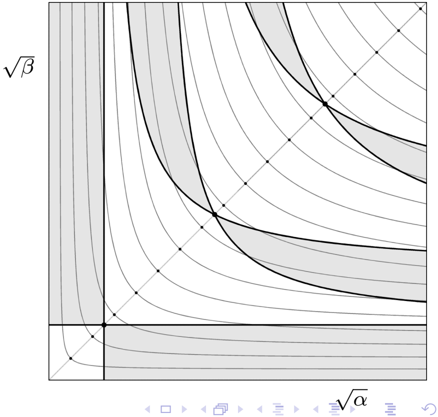


4-Point Non-Selfadjoint Operator

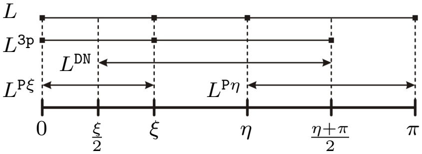


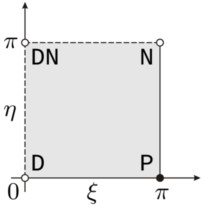




- $\sigma(L) = \sigma(L^{P\xi}) \cup \sigma(L^{P\eta}) \cup \sigma(L^{DN})$,
- $\Sigma(L) \overset{???}{=} \Sigma(L^{P\xi}) \cup \Sigma(L^{P\eta}) \cup \Sigma(L^{DN})$,
- $\sigma(L^{3p}) = \sigma(L^{DN})$,
- $\Sigma(L^{P\xi}) = \Sigma(L^{P\eta})$ for $\eta = \pi - \xi$.

4-Point Non-Selfadjoint Operator





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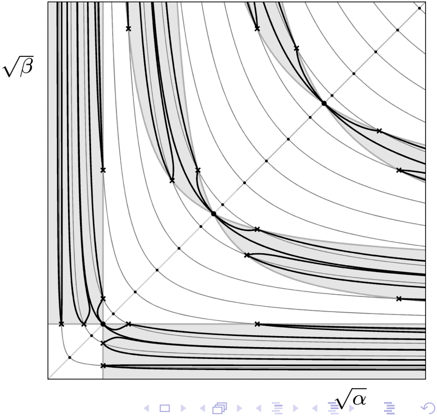
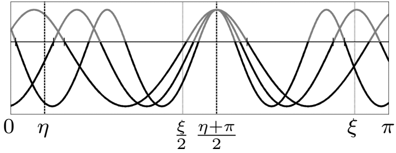
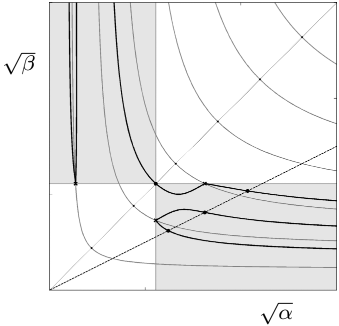
- $\Sigma(L) \overset{???}{=} \Sigma(L^{\mathrm{P}\xi}) \cup \Sigma(L^{\mathrm{P}\eta}) \cup \Sigma(L^{\mathrm{DN}})$,
- $\sigma(L^{3\mathrm{p}}) = \sigma(L^{\mathrm{DN}})$,
- $\Sigma(L^{\mathrm{P}\xi}) = \Sigma(L^{\mathrm{P}\eta})$ for $\eta = \pi - \xi$.



Non-Selfadjoint Operator $L^{3\mathrm{p}}$





Proposition

The Fučík spectrum of the operator $L^{3\mathrm{p}}$ is given by

$$\Sigma\left(L^{3\mathrm{p}}\right) = \bigcup_{l\in\mathbb{N}_0} \mathcal{C}_l^{3\mathrm{p}}, \qquad \textit{where} \begin{cases} \mathcal{C}_0^{3\mathrm{p}} := \mathcal{C}_0^{\mathrm{P}\xi}, \\ \mathcal{C}_l^{3\mathrm{p}} := \bigcup_{k\in\mathbb{N}_0} \left(\mathcal{C}_{k,l}^{3\mathrm{p}+} \cup \mathcal{C}_{k,l}^{3\mathrm{p}-}\right), \quad l \in \mathbb{N}, \end{cases}$$

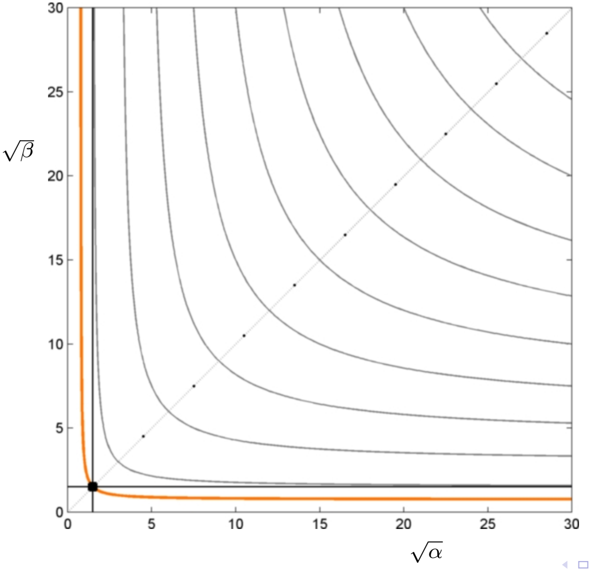
and where we define for $k, l \in \mathbb{N}_0$

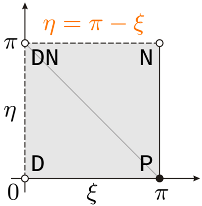
$$\mathcal{C}_{k,l}^{3\mathrm{p}-} := \left\{(\alpha,\beta) \in S_k^{\mathrm{P}} \cap S_l^{\mathrm{DN}} : (\beta,\alpha) \in \mathcal{C}_{k,l}^{3\mathrm{p}+}\right\},$$

$$\mathcal{C}_{k,l}^{3\mathrm{p}+} := \left\{(\alpha,\beta) \in S_k^{\mathrm{P}} \cap S_l^{\mathrm{DN}} : \begin{cases} \frac{\eta+\pi}{2\pi} = F_{k,l}(\alpha,\beta) & \text{for } k \text{ even}, \\ \frac{\eta+\pi}{2\pi} = F_{k,l}(\beta,\alpha) & \text{for } k \text{ odd}, \end{cases}\right\},$$

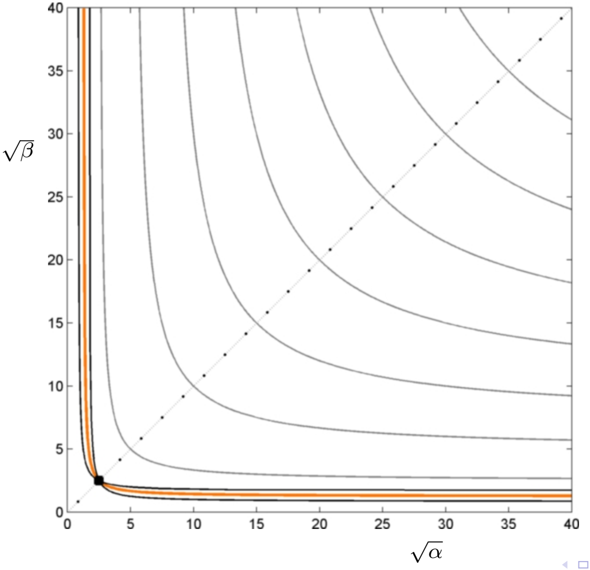
$$F_{k,l}(\alpha,\beta) := \frac{\xi}{\pi}\frac{\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}} + \frac{l-k}{2\sqrt{\alpha}} + \frac{l+k+1}{2\sqrt{\beta}}.$$

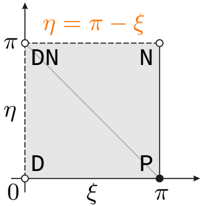
Non-Selfadjoint Operator L^{3p}



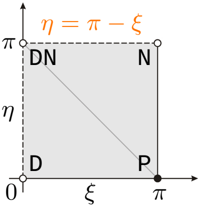


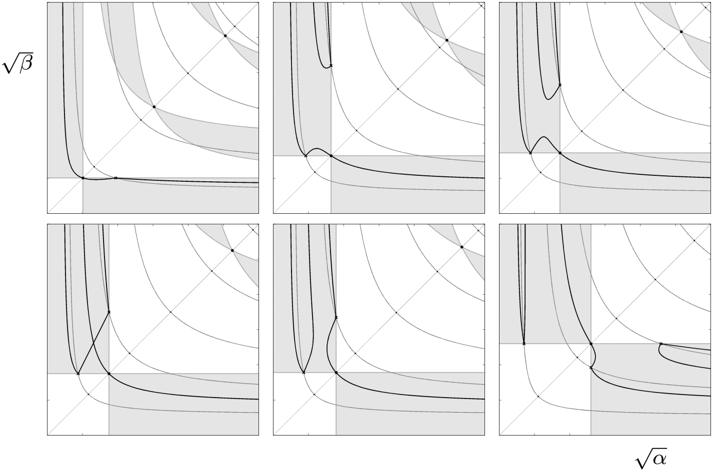
Non-Selfadjoint Operator $L^{3\mathrm{p}}$





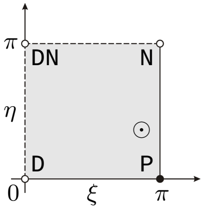
Non-Selfadjoint Operator L^{3p}

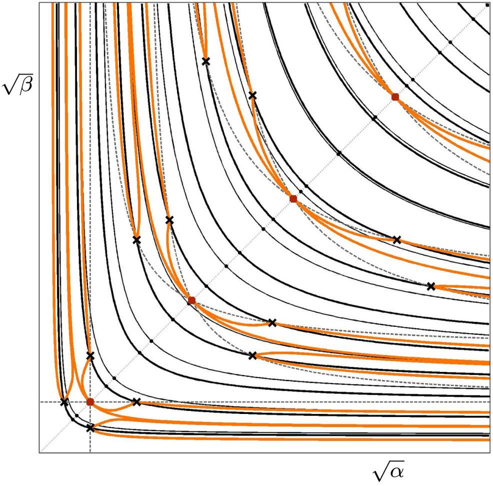




Conclusion

$$\begin{cases} -u''(x) = \alpha u^+(x) - \beta u^-(x), & x \in [0,\pi], \\ u'(0) = u'(\xi), \quad u(\eta) = u(\pi), & \xi \in (0,\pi], \quad \eta \in [0,\pi). \end{cases} \qquad (3)$$





Lemma

The Fučík spectrum of the four-point problem (3) *is given by*

$$\Sigma(L) = \Sigma\big(L^{\mathrm{P}\xi}\big) \cup \Sigma\big(L^{\mathrm{P}\eta}\big) \cup \Sigma\big(L^{\mathrm{3p}}\big).$$

Remark

The Fučík spectrum of the Dirichlet–Neumann operator $\Sigma(L^{\mathrm{DN}})$ determines the intersection of $\Sigma(L^{\mathrm{P}\xi})$ and $\Sigma(L^{\mathrm{3p}})$,

$$\Sigma\big(L^{\mathrm{P}\xi}\big) \cap \Sigma\big(L^{\mathrm{3p}}\big) \subset \Sigma\big(L^{\mathrm{DN}}\big).$$

The Operator L and the Adjoint Operator L^*

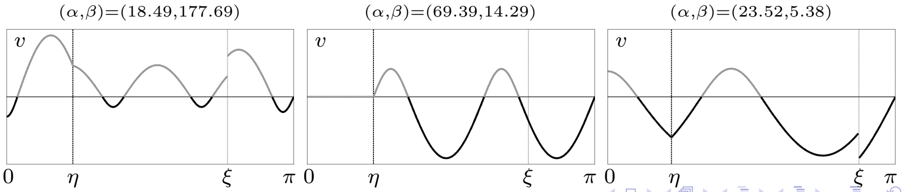
Definition

Let us define $L : \mathrm{dom}(L) \subset W^{2,2}(0,\pi) \to L^2(0,\pi) : u \mapsto -u''$,

$$\begin{aligned}\mathrm{dom}(L) := \{u \in AC(0,\pi) : \quad & u' \in AC(0,\pi),\ u'' \in L^2(0,\pi), \\ & u'(0) = u'(\xi),\ u(\eta) = u(\pi)\},\end{aligned}$$

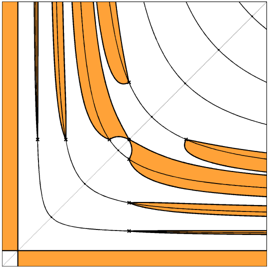
and $L^* : \mathrm{dom}(L^*) \subset L^2(0,\pi) \to W^{-2,2}(0,\pi) : u \mapsto -u''$,

$$\begin{aligned}\mathrm{dom}(L^*) := \{v \in L^2(0,\pi) : \quad & v,\ v' \text{ are absolutely continuous on } (0,\eta),\ (\eta,\xi),\ (\xi,\pi), \\ & v'(0) = 0,\ v(\pi) = 0, \\ & v(\xi+) = v(\xi-) - v(0), \quad v'(\xi+) = v'(\xi-), \\ & v(\eta+) = v(\eta-), \qquad\qquad v'(\eta+) = v'(\eta-) + v'(\pi)\}.\end{aligned}$$



Solvability

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = f(x), & x \in (0,\pi), \\ u'(0) = u'(\xi), \quad u(\eta) = u(\pi), & \xi,\eta \in (0,\pi). \end{cases} \tag{4}$$



Definition

Region of type I is a continuous component $R^{\mathrm{I}} \subset \mathbb{R}^2 \setminus \Sigma(L)$ such that

$$\exists (\lambda,\lambda) \in \overline{R^{\mathrm{I}}} : \lambda \notin \sigma(L).$$

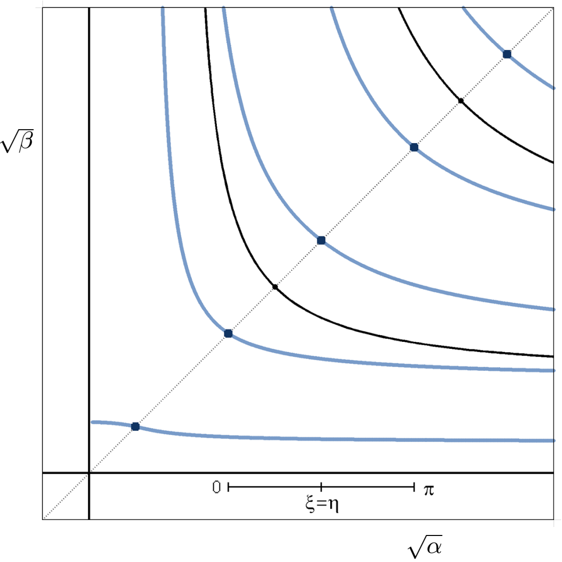
Region of type II is a continuous component $R^{\mathrm{II}} \subset \mathbb{R}^2 \setminus \Sigma(L)$ such that

$$\forall (\lambda,\lambda) \in \overline{R^{\mathrm{II}}} : \lambda \in \sigma(L).$$

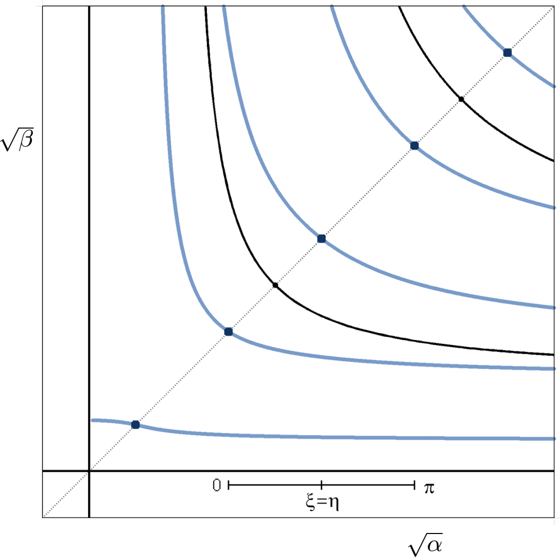
Theorem

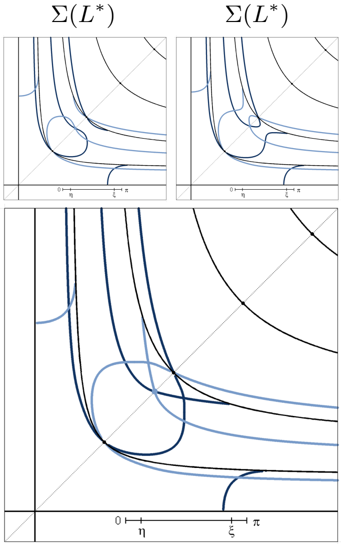
1. *Let $(\alpha,\beta) \in R^{\mathrm{I}}$, where R^{I} is a **region of type** I.*
   Then the problem (4) has a solution for every $f = f(x) \in L^2(0,\pi)$.
2. *Let $(\alpha,\beta) \in R^{\mathrm{II}}$, where R^{II} is a **region of type** II.*
   Then there exists a function $f = f(x) \in L^2(0,\pi)$ such that the problem (4) has no solution.

The Fučík Spectrum of L^* - Numerical Experiment

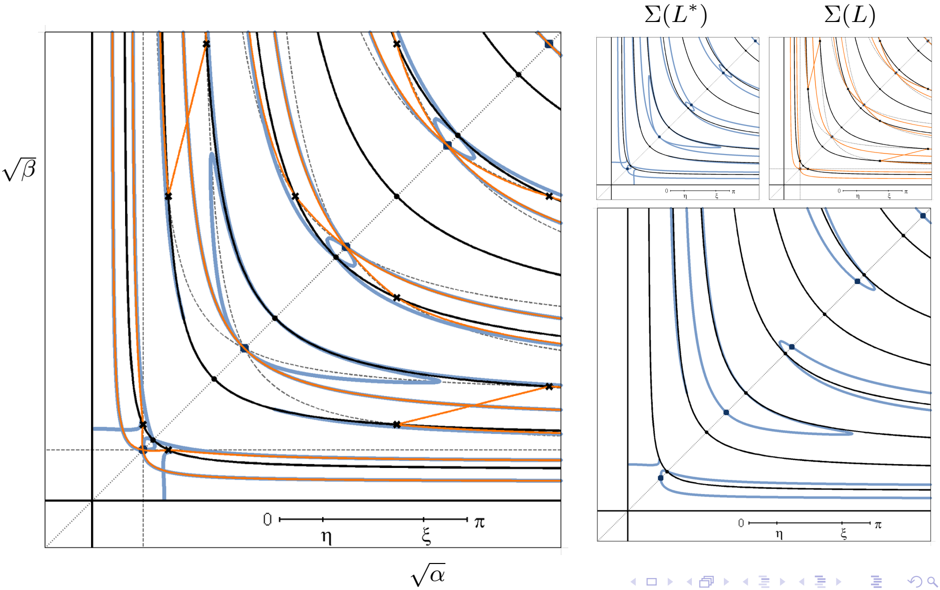


The Fučík Spectrum of L^* - Numerical Experiment

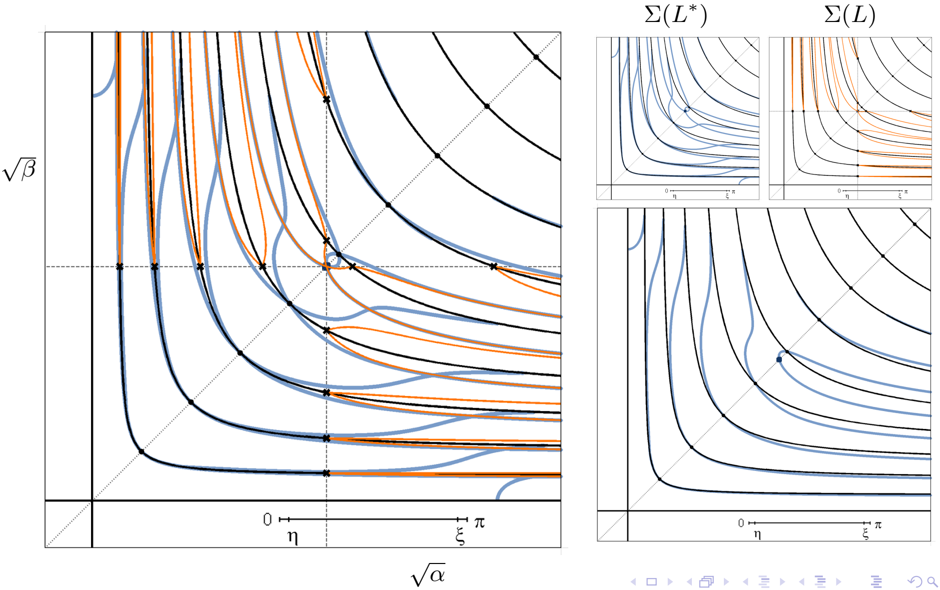




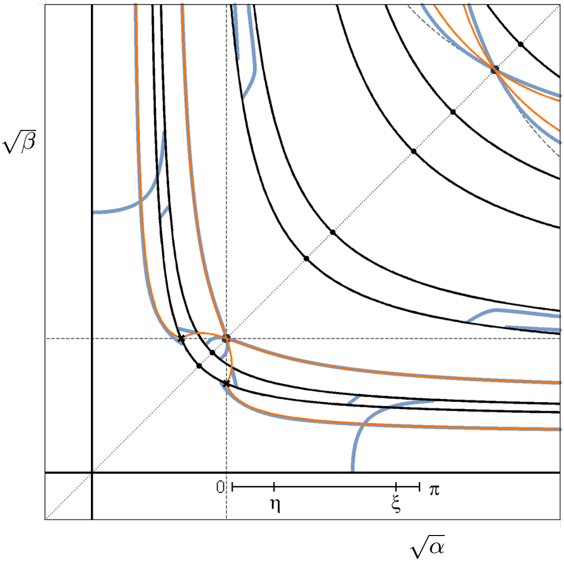
Connection between the Fučík Spectra of L and L^*

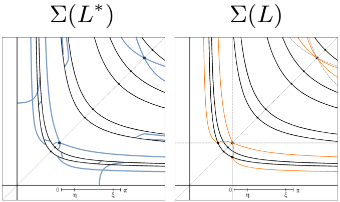


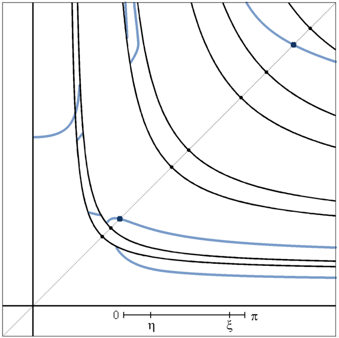
Connection between the Fučík Spectra of L and L^*



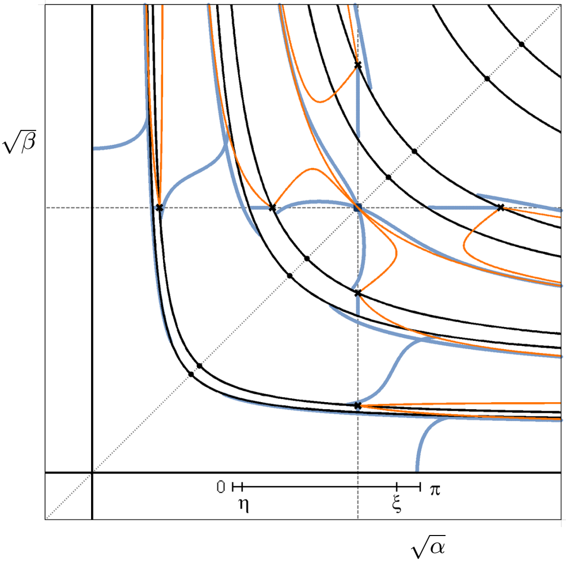
Connection between the Fučík Spectra of L and L^*

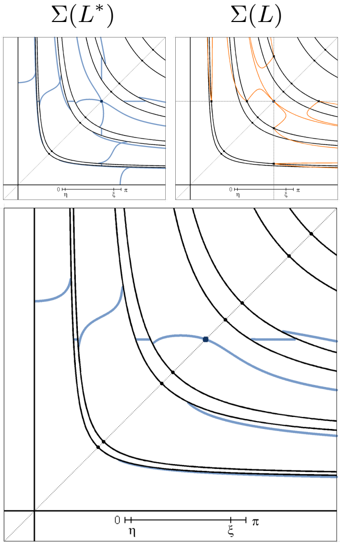






Connection between the Fučík Spectra of L and L^*





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