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The Fučík Spectrum Structure:
Known Results, Experiments and Open Problems

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Boundary Value Problems and Related Topics

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Outline

The Fučík Spectrum

$$\Sigma(L) := \left\{(\alpha, \beta) \in \mathbb{R}^2 : \; Lu = \alpha u^+ - \beta u^- \text{ has a nontrivial solution}\right\},$$

where

$$u^+ := \max\{u, 0\}, \quad u^- := \max\{-u, 0\}.$$

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- $$Lu = (\lambda + \varepsilon\eta)u + \varepsilon|u|$$
- Let $\lambda \in \sigma_d(L) = \sigma(L)$. For $\varepsilon \neq 0$, we have

$$\left\{ \begin{array}{c} u = Pu + \varepsilon(L - \lambda I)^{-1}\left[(I-P)(|u| + \eta u)\right] + P(|u| + \eta u), \\ \|u\|^2 = 1. \end{array} \right. \qquad (1)$$

The Local Existence of $\Sigma(L)$ Close to the Diagonal

Definition

Let us define the functional $\mathcal{P} : \mathrm{dom}(\mathcal{P}) \subset L^2(\Omega) \to \mathbb{R}$

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Theorem (Ben-Naoum, Fabry and Smets)

Let $\lambda \in \sigma_d(L)$. Assume that $u \neq 0$ a.e. in Ω for all $u \in \ker(L - \lambda I) \setminus \{0\}$ and that the functional $\mathcal{P}$, restricted to the unit sphere, has a stationary point z_0. Moreover, let us assume that the following nondegeneracy condition is satisfied at this stationary point z_0

$$y \in \ker(L - \lambda I),\ \langle z_0, y \rangle = 0,\ P(\mathrm{sgn}(z_0)y) = \langle |z_0|, z_0 \rangle\, y \quad \Rightarrow \quad y = 0. \qquad \text{(ND1)}$$

Then, there exists a neighborhood $\mathcal{U}(0) \subset \mathbb{R}$ of 0 and two continuous mappings $\eta : \mathcal{U}(0) \to \mathbb{R}$ and $u : \mathcal{U}(0) \to \mathrm{dom}(L) \subset L^2(\Omega)$ such that

$$\begin{aligned} u(0) &= z_0, \\ \eta(0) &= -\langle |z_0|, z_0 \rangle, \\ Lu(\varepsilon) &= \varepsilon |u(\varepsilon)| + (\lambda + \varepsilon \eta(\varepsilon)) u(\varepsilon), \quad \|u(\varepsilon)\| = 1 \qquad \text{for } \varepsilon \in \mathcal{U}(0). \end{aligned}$$

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For $\dim \ker(L - \lambda I) \geq 2$, (ND1) can be replaced by

$$\exists \mathcal{V}(z_0) \subset S \text{ such that } \begin{cases} \max \{\mathcal{P}(z) : \ z \in \mathcal{V}(z_0)\} = \mathcal{P}(z_0), & \mathcal{P}(z) < \mathcal{P}(z_0) \quad \forall z \in \partial \mathcal{V}(z_0), \\ \min \{\mathcal{P}(z) : \ z \in \mathcal{V}(z_0)\} = \mathcal{P}(z_0), & \mathcal{P}(z) > \mathcal{P}(z_0) \quad \forall z \in \partial \mathcal{V}(z_0), \end{cases} \tag{ND1'}$$

where S is a unit sphere in $\ker(L - \lambda I)$.

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The Existence of $\Sigma(L)$ Away from the Diagonal

Theorem (Ben-Naoum, Fabry and Smets)

Let $E \subset \mathbb{R}$ *and* u_0, $\alpha_0 \neq \beta_0$ *be such that* $(\alpha_0, \beta_0) \in (E \times E) \cap \Sigma(L)$, $Lu_0 = \alpha_0 u_0^+ - \beta_0 u_0^-$, $\|u_0\| = 1$. *Moreover, let*

- *the equation* $Lu = \lambda u$ *has no solution of a constant sign for* $\lambda \in E$,
- *the nondegeneracy condition (ND2) holds for* $(\alpha, \beta) = (\alpha_0, \beta_0)$,
- $\mathrm{dom}(L) \subset L^p(\Omega)$ *for some* $p > 2$ *and the injection is continuous when* $\mathrm{dom}(L)$ *is equipped with the graph norm.*

The Existence of $\Sigma(L)$ Away from the Diagonal

Nondegeneracy Condition:

For any $u \neq 0$ verifying $Lu = \alpha u^+ - \beta u^-$, we have
$u \neq 0$ a.e. on Ω and $\dim\ker\left[L - (\alpha\chi_{\{u>0\}} + \beta\chi_{\{u<0\}})I\right] = 1.$ (ND2)

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Then, there exist neighborhoods $\mathcal{A}(\alpha_0) \subset \mathbb{R}$, $\mathcal{B}(\beta_0) \subset \mathbb{R}$, $\mathcal{U}(u_0) \subset \mathrm{dom}(L)$ *and continuous mappings* $\beta = \beta(\alpha) : \mathcal{A}(\alpha_0) \to \mathcal{B}(\beta_0)$, *and* $u = u(\alpha) : \mathcal{A}(\alpha_0) \to \mathcal{U}(u_0)$ *such that*

- $\beta(\alpha_0) = \beta_0$ *and* $u(\alpha_0) = u_0$,
- $Lu(\alpha) = \alpha u^+(\alpha) - \beta(\alpha)u^-(\alpha)$ *with* $\|u(\alpha)\| = 1$ *for* $\alpha \in \mathcal{A}(\alpha_0)$,
- $Lu = \alpha u^+ - \beta u^-$ *with* $\|u\| = 1$ *and* $u \in \mathcal{U}(u_0)$, $\alpha \in \mathcal{A}(\alpha_0)$, $\beta \in \mathcal{B}(\beta_0) \Rightarrow$ $u = u(\alpha)$ *and* $\beta = \beta(\alpha)$.

Moreover, the function $\beta = \beta(\alpha)$ *is differentiable at* α_0 *and* $\beta'(\alpha_0) = -\dfrac{\|u_0^+\|^2}{\|u_0^-\|^2} < 0.$

Outline

Example 1

$$\begin{cases} u^{\mathrm{IV}}(x) + (m^2+n^2)u''(x) = \alpha u^+(x) - \beta u^-(x), \quad x \in (0,\pi), \\ u(0) = u''(0) = u(\pi) = u''(\pi) = 0. \end{cases} \tag{2}$$

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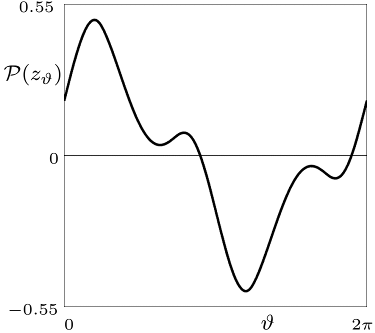
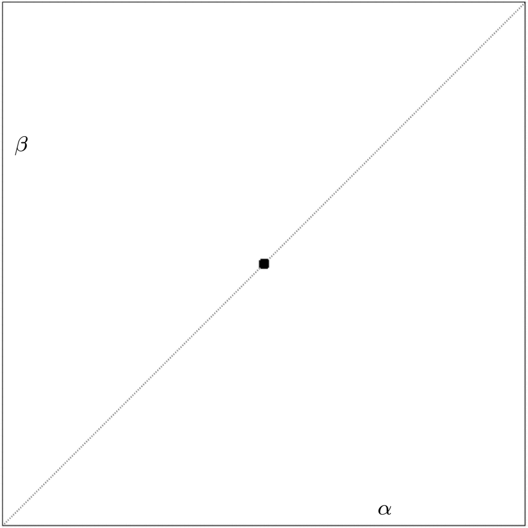
$$\alpha = \beta = -m^2 n^2, \quad m = 5, n = 9,$$

$$z_\vartheta(x) := \varphi_1(x)\cos\vartheta + \varphi_2(x)\sin\vartheta,$$

$$\vartheta \in [0, 2\pi).$$





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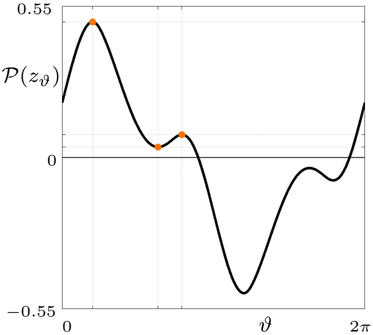

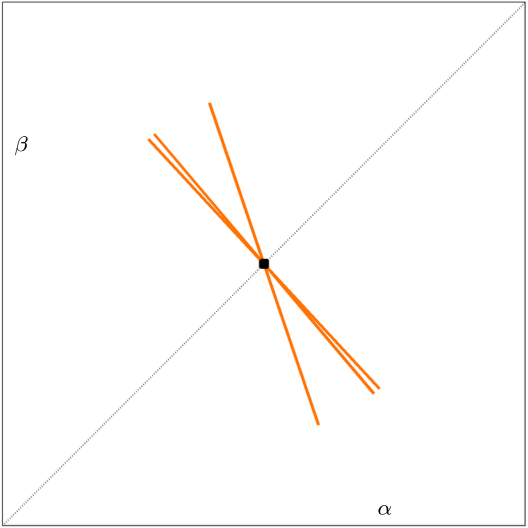

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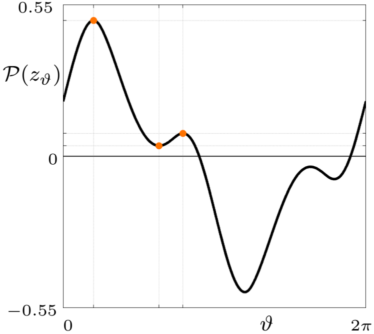

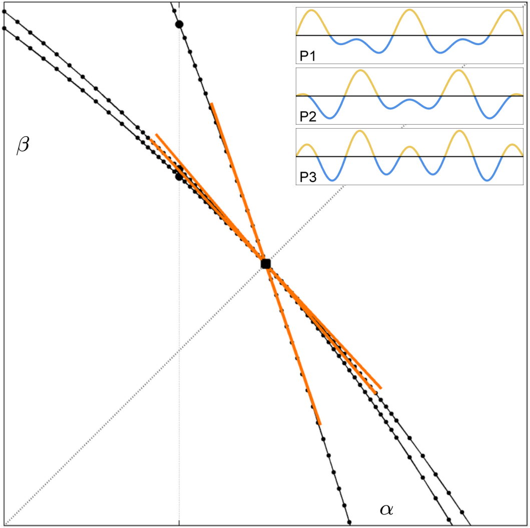

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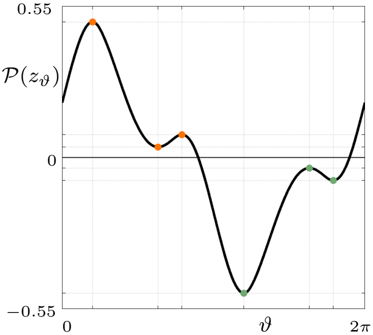

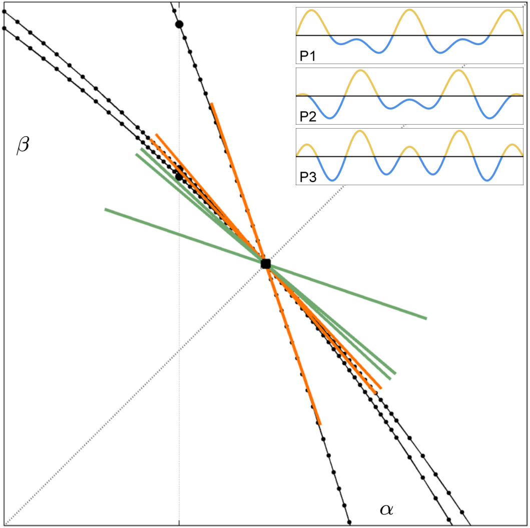

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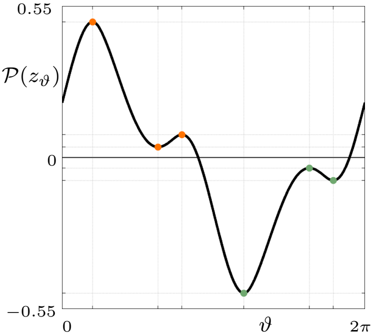
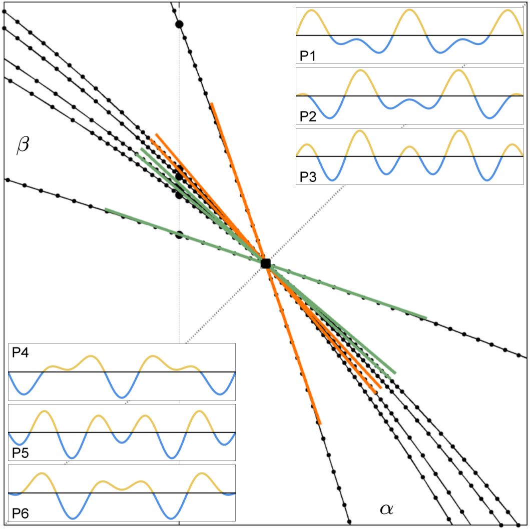
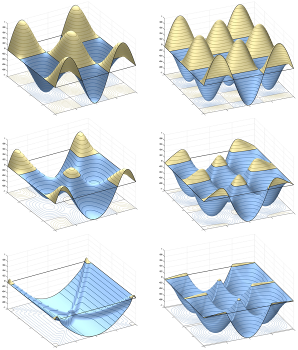
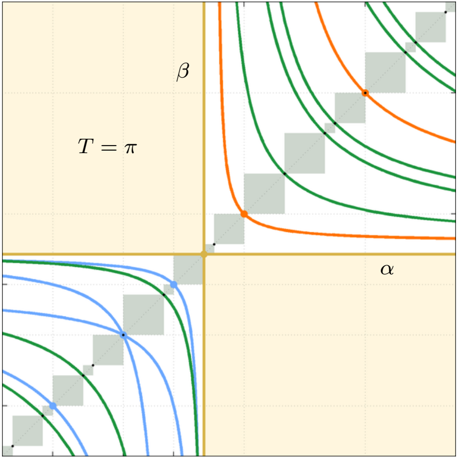




Example 2

$$\begin{cases} -(u_{tt}(x,t) - u_{xx}(x,t)) = \alpha u^{+}(x,t) - \beta u^{-}(x,t), & (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0, & t \in \mathbb{R}, \\ u(x,t) = u(x,t+T), & (x,t) \in (0,\pi) \times \mathbb{R}, \end{cases} \tag{3}$$





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$$\alpha = \beta = -9,$$

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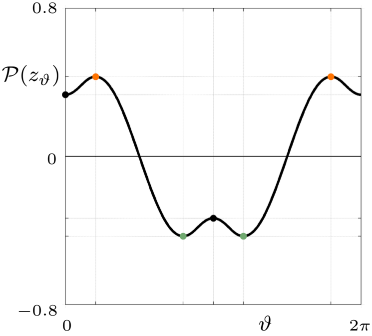

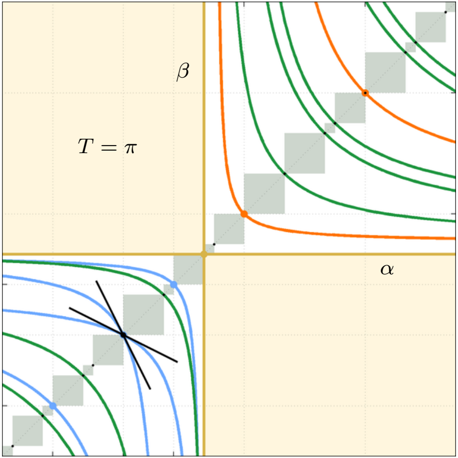

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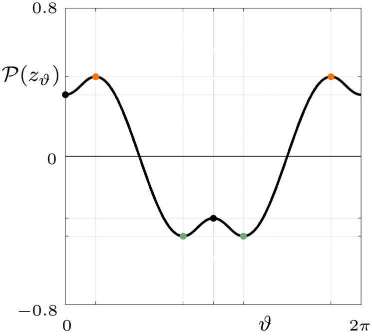
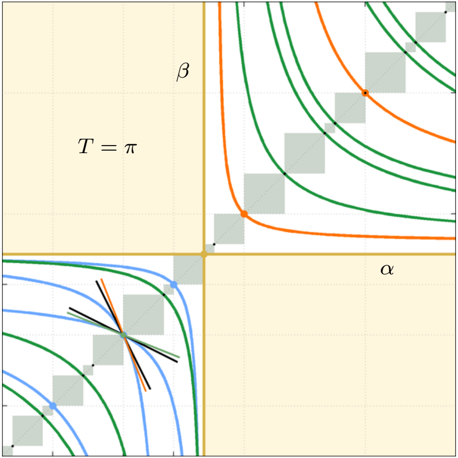
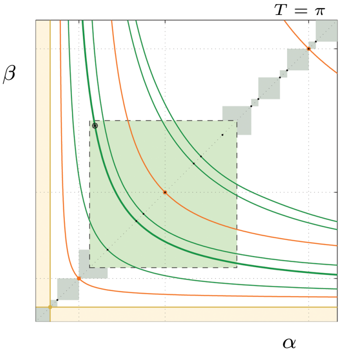




Verification of Explored Fučík Curves

- $E \subset \mathbb{R}, \quad E \cap \sigma(L) = \{\lambda_1, \dots, \lambda_p\}$
- $Z := \oplus_{i=1}^{p} \ker(L - \lambda_i I)$
- P – orthogonal projection onto Z



Lemma

Let $(\alpha, \beta) \in E \times E$, $\alpha \neq \beta$.
$(\alpha, \beta) \in \Sigma(L) \quad \Longleftrightarrow \quad \exists\, u \in L^2(\Omega)$ *such that* $Pu \neq 0$ *and that*

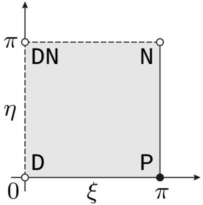
$$\left\{ \begin{array}{ll} \langle |u|, \varphi_i \rangle = 0 & \text{for } \langle u, \varphi_i \rangle = 0, \\ \dfrac{\langle |u|, \varphi_i \rangle}{\langle u, \varphi_i \rangle} = \dfrac{\lambda_i - \frac{\alpha+\beta}{2}}{\frac{\alpha-\beta}{2}} & \text{for } \langle u, \varphi_i \rangle \neq 0, \end{array} \right\}$$

for every eigenfunction $\varphi_i \in \ker(L - \lambda_i I)$, $i = 1, \dots, p$.

Outline

Four-Point BVP

$$\begin{cases} -u''(x) = \alpha u^+(x) - \beta u^-(x), & x \in (0,\pi), \\ u'(0) = u'(\xi), \quad u(\eta) = u(\pi), & \xi \in (0,\pi), \quad \eta \in (0,\pi). \end{cases} \tag{4}$$



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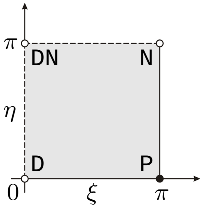
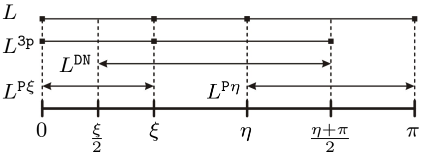




Definition

Let us define the following operators

$$Lu := L^{\mathtt{P}\xi} u := L^{\mathtt{P}\eta} u := L^{\mathtt{DN}} u := L^{\mathtt{3p}} u := -u''$$

for $\xi, \eta \in (0,\pi)$ by

$$D(L) := \{u \in C^2([0,\pi]) : \quad u'(0) = u'(\xi), \quad u(\eta) = u(\pi) \},$$

$$D\left(L^{\mathtt{P}\xi}\right) := \{u \in C^2([0,\pi]) : \quad u'(0) = u'(\xi), \quad u(0) = u(\xi) \},$$

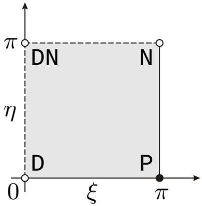
$$D\left(L^{\mathtt{P}\eta}\right) := \{u \in C^2([0,\pi]) : \quad u'(\eta) = u'(\pi), \quad u(\eta) = u(\pi) \},$$

$$D\left(L^{\mathtt{DN}}\right) := \{u \in C^2([0,\pi]) : \quad u\left(\tfrac{\xi}{2}\right) = 0, \quad u'\left(\tfrac{\eta+\pi}{2}\right) = 0 \},$$

$$D\left(L^{\mathtt{3p}}\right) := \{u \in C^2([0,\pi]) : \quad u'(0) = u'(\xi), \quad u'\left(\tfrac{\eta+\pi}{2}\right) = 0, \quad u(0)u(\xi) \le 0\}.$$

Four-Point BVP

$$\begin{cases} -u''(x) = \alpha u^+(x) - \beta u^-(x), & x \in (0,\pi), \\ u'(0) = u'(\xi), \quad u(\eta) = u(\pi), & \xi \in (0,\pi), \quad \eta \in (0,\pi). \end{cases} \tag{4}$$



Remark

The spectra of $L^{\mathtt{P}\xi}$, $L^{\mathtt{P}\eta}$ and $L^{\mathtt{DN}}$ are pure point discrete spectra made only of the following real eigenvalues

$$\lambda_k^{\mathtt{P}\xi} := \left(\frac{2k\pi}{\xi}\right)^2, \ \lambda_m^{\mathtt{P}\eta} := \left(\frac{2m\pi}{\pi-\eta}\right)^2 \quad \textit{and} \quad \lambda_l^{\mathtt{DN}} := \left(\frac{(2l+1)\pi}{\pi+\eta-\xi}\right)^2, \ k,l,m \in \mathbb{N}_0,$$

and

$$\sigma(L) = \sigma(L^{\mathtt{P}\xi}) \cup \sigma(L^{\mathtt{P}\eta}) \cup \sigma(L^{\mathtt{DN}}).$$

Four-Point BVP

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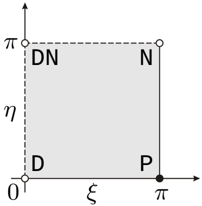

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The spectra of $L^{\mathrm{P}\xi}$, $L^{\mathrm{P}\eta}$ and L^{DN} are pure point discrete spectra made only of the following real eigenvalues

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- $\sigma(L^{3\mathrm{p}}) = \sigma(L^{\mathrm{DN}})$,
- $\sigma(L^{\mathrm{P}\xi}) = \sigma(L^{\mathrm{P}\eta})$ for $\eta = \pi - \xi$.

Four-Point BVP

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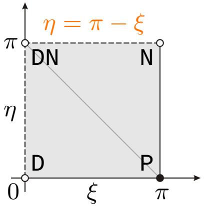

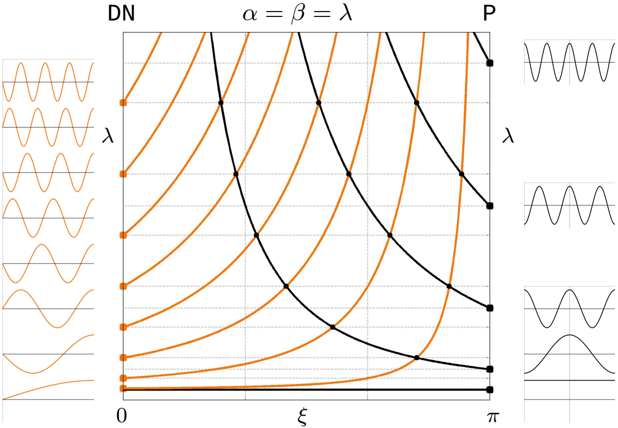

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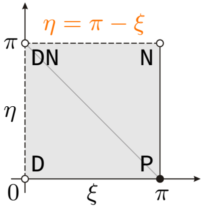
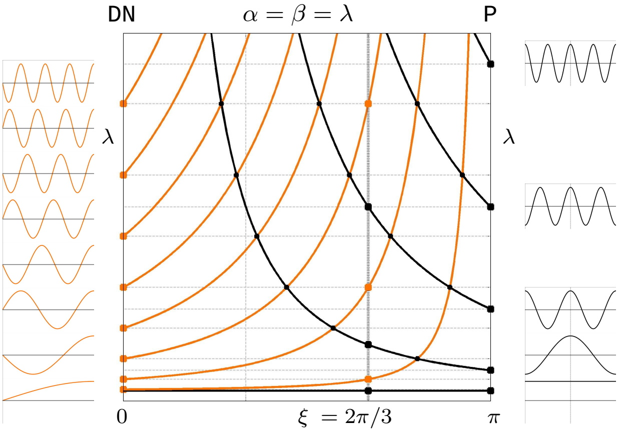
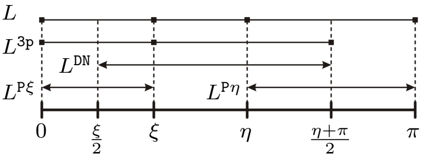


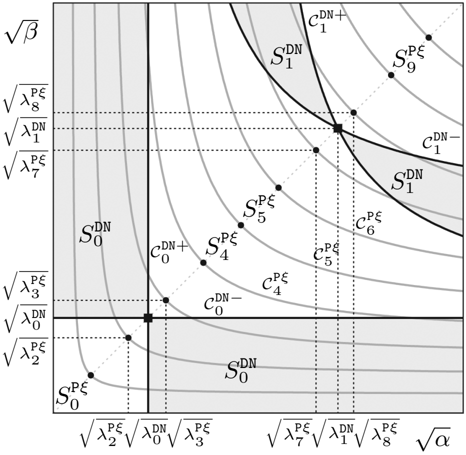


Outline

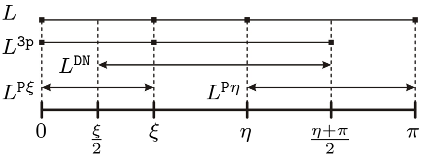
Construction of the Fučík Spectrum



- $\sigma(L) = \sigma(L^{\mathrm{P}\xi}) \cup \sigma(L^{\mathrm{P}\eta}) \cup \sigma(L^{\mathrm{DN}})$,



Construction of the Fučík Spectrum



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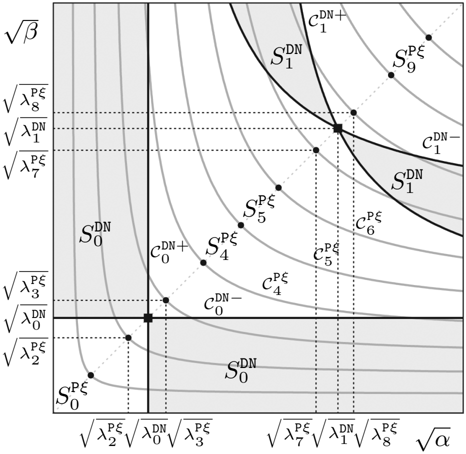

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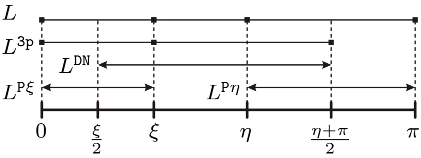

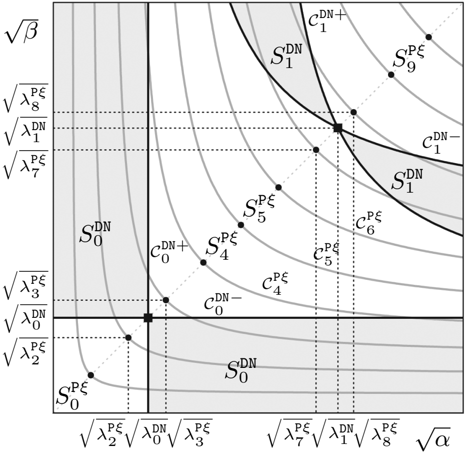

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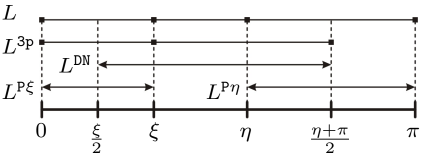
Construction of the Fučík Spectrum



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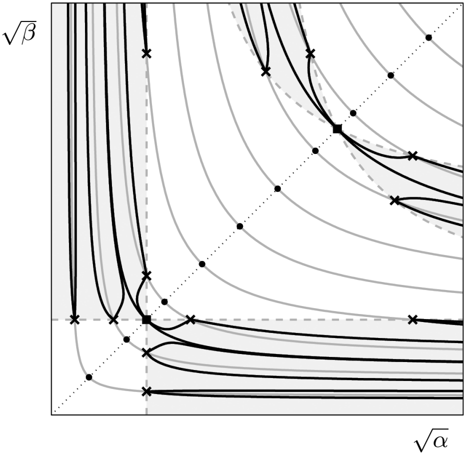

Construction of the Fučík Spectrum

Theorem

The Fučík spectrum of the four-point problem (4) is given by

$$\Sigma(L) = \Sigma(L^{\mathtt{P}\xi}) \cup \Sigma(L^{\mathtt{P}\eta}) \cup \Sigma(L^{\mathtt{3p}}).$$

Construction of the Fučík Spectrum

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The Fučík spectrum of the four-point problem (4) is given by

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Proof

- $\Sigma(L) \subset \Sigma(L^{\mathtt{P}\xi}) \cup \Sigma(L^{\mathtt{P}\eta}) \cup \Sigma(L^{\mathtt{3p}})$
- $\Sigma(L) \supset \Sigma(L^{\mathtt{P}\xi}) \cup \Sigma(L^{\mathtt{P}\eta}) \cup \Sigma(L^{\mathtt{3p}})$

Construction of the Fučík Spectrum

Theorem

The Fučík spectrum of the four-point problem (4) is given by

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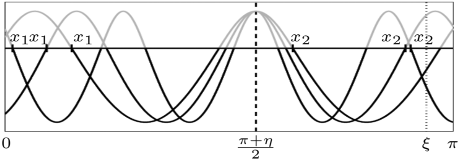
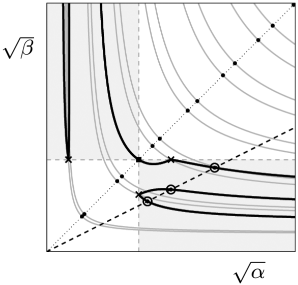
Proof

- $\Sigma(L) \subset \Sigma(L^{\mathrm{P}\xi}) \cup \Sigma(L^{\mathrm{P}\eta}) \cup \Sigma(L^{3\mathrm{p}})$
- $\Sigma(L) \supset \Sigma(L^{\mathrm{P}\xi}) \cup \Sigma(L^{\mathrm{P}\eta}) \cup \Sigma(L^{3\mathrm{p}})$
- Let $\alpha, \beta > 0$ and $u \in C^2([0, \pi])$ be the nontrivial solution of $-u'' = \alpha u^+ - \beta u^-$ on $(0, \pi)$. Then the following statements hold ($x_1, x_2 \in [0, \pi]$):
  - If $u(x_1) = u(x_2)$ then $u'(x_1) = u'(x_2)$ or $u'(x_1) = -u'(x_2)$. (5)
  - $u(x_1) = u(x_2)$ and $u'(x_1) = -u'(x_2)$ if and only if $u'(\frac{x_1+x_2}{2}) = 0$. (6)
  - If $u'(x_1) = u'(x_2)$ and $u(x_1)u(x_2) > 0$ then $u(x_1) = u(x_2)$. (7)

■

Construction of $\Sigma(L^{3\mathrm{p}})$





Proposition

The Fučík spectrum of $L^{3\mathrm{p}}$ on $\mathbb{R}^+ \times \mathbb{R}^+$ is given by

$$\Sigma\left(L^{3\mathrm{p}}\right) = \bigcup_{l\in\mathbb{N}_0} \mathcal{C}_l^{3\mathrm{p}}, \qquad \text{where } \mathcal{C}_l^{3\mathrm{p}} := \bigcup_{k\in\mathbb{N}_0}\left(\mathcal{C}_{k,l}^{3\mathrm{p}+} \cup \mathcal{C}_{k,l}^{3\mathrm{p}-}\right), \quad l \in \mathbb{N}_0,$$

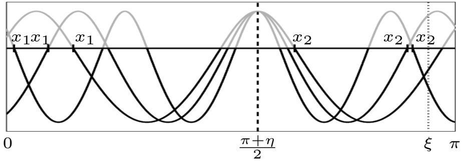
and where we define for $k, l \in \mathbb{N}_0$

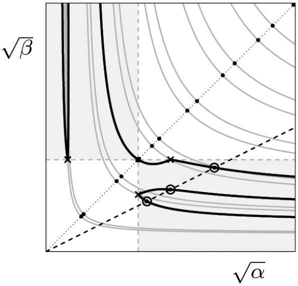
$$\mathcal{C}_{k,l}^{3\mathrm{p}-} := \left\{(\alpha,\beta) \in S_k^{\mathrm{P}\xi} \cap S_l^{\mathrm{DN}} : (\beta,\alpha) \in \mathcal{C}_{k,l}^{3\mathrm{p}+}\right\},$$

$$\mathcal{C}_{k,l}^{3\mathrm{p}+} := \left\{(\alpha,\beta) \in S_k^{\mathrm{P}\xi} \cap S_l^{\mathrm{DN}} : \begin{cases} \frac{\pi+\eta}{2\pi} = F_{k,l}(\alpha,\beta) & \text{for } k+l \text{ even} \\ \frac{\pi+\eta}{2\pi} = F_{k,l}(\beta,\alpha) & \text{for } k+l \text{ odd} \end{cases}\right\},$$

$$F_{k,l}(\alpha,\beta) := \frac{\xi}{\pi}\frac{\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}} + \frac{l-k}{2\sqrt{\alpha}} + \frac{l+k+1}{2\sqrt{\beta}}.$$

Construction of $\Sigma(L^{3\mathrm{p}})$





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$$\mathcal{C}_{k,l}^{3\mathrm{p}+} := \left\{(\alpha, \beta) \in \right.$$

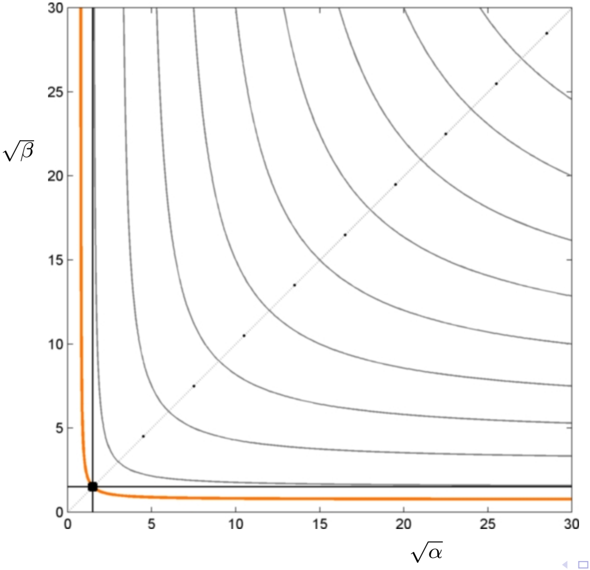
$$F_{k,l}(\alpha, \beta) := \frac{\xi}{\pi} \frac{\sqrt{\beta}}{\sqrt{\alpha} + \sqrt{}}$$

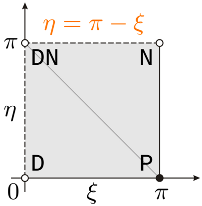
$$\mathcal{C}_l^{3\mathrm{p}} = \bigcup_{k=k_l^{\mathtt{min}}}^{k_l^{\mathtt{max}}-1} \left(\mathcal{C}_{k,l}^{3\mathrm{p}+} \cup \mathcal{C}_{k,l}^{3\mathrm{p}-}\right), \quad l \in \mathbb{N}_0,$$

where

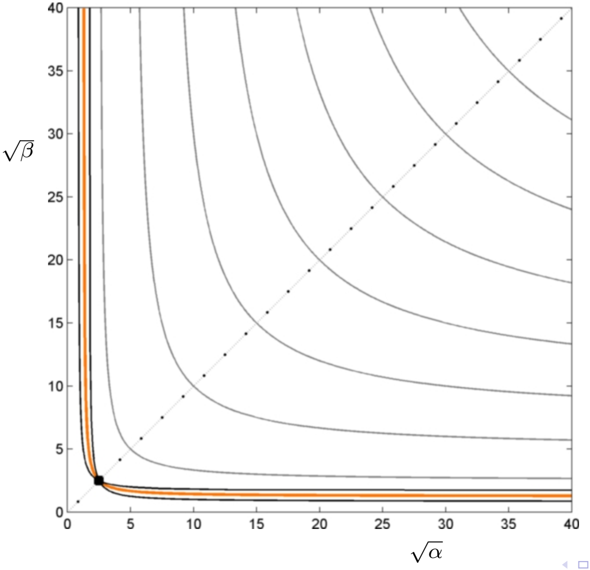
$$k_l^{\mathtt{min}} := \left\lfloor \frac{l\xi}{\pi + \eta - \xi} \right\rfloor \quad \text{and} \quad k_l^{\mathtt{max}} := \left\lceil \frac{(l+1)\xi}{\pi + \eta - \xi} \right\rceil.$$

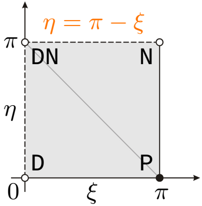
Qualitative Properties of $\Sigma(L^{3p})$





Qualitative Properties of $\Sigma(L^{3\mathrm{p}})$





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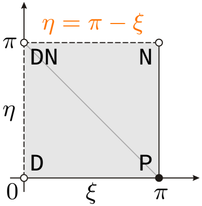
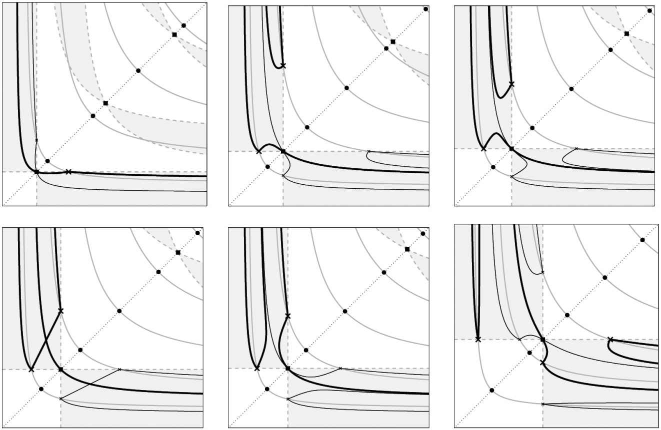
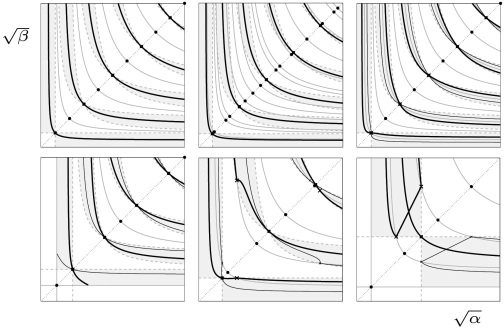


$\sqrt{\beta}$



$\sqrt{\alpha}$

Conclusion of the Construction



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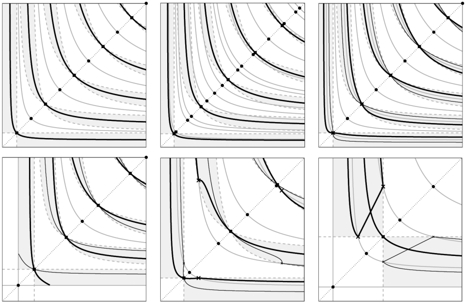


Remark

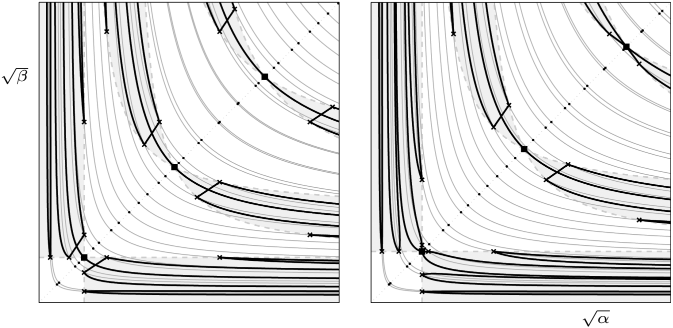
- Since $\sigma(L^{3p}) = \sigma(L^{DN})$ but $\Sigma(L^{3p}) \neq \Sigma(L^{DN})$, we conclude that

$$\begin{array}{rclcl} \sigma(L) & = & \sigma(L^{P\xi}) \cup \sigma(L^{P\eta}) \cup \sigma(L^{3p}) & = & \sigma(L^{P\xi}) \cup \sigma(L^{P\eta}) \cup \sigma(L^{DN}), \\ \Sigma(L) & = & \Sigma(L^{P\xi}) \cup \Sigma(L^{P\eta}) \cup \Sigma(L^{3p}) & \neq & \Sigma(L^{P\xi}) \cup \Sigma(L^{P\eta}) \cup \Sigma(L^{DN}). \end{array}$$

- The Fučík spectrum $\Sigma(L^{DN})$ determines the intersection of $\Sigma(L^{P\xi})$ and $\Sigma(L^{3p})$:

$$\Sigma(L^{3p}) \cap \Sigma(L^{P\xi}) = \Sigma(L^{DN}) \cap \Sigma(L^{P\xi}) \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^+.$$

Conclusion of the Construction



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- Since $\sigma(L^{3p}) = \sigma(L^{DN})$ but $\Sigma(L^{3p}) \neq \Sigma(L^{DN})$, we conclude that

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Outline

The Operator L and the Adjoint Operator L^*

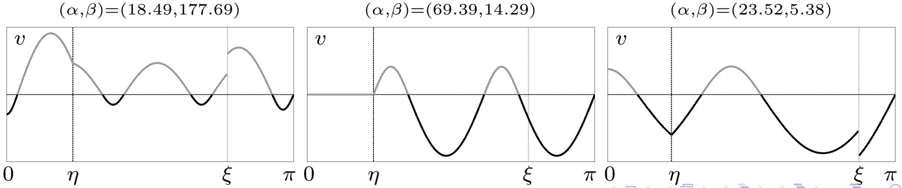
Definition

Let us define $L : \mathrm{dom}(L) \subset W^{2,2}(0,\pi) \to L^2(0,\pi) : u \mapsto -u''$,

$$\mathrm{dom}(L) := \{u \in AC(0,\pi) : \quad u' \in AC(0,\pi),\ u'' \in L^2(0,\pi), \\ u'(0) = u'(\xi),\ u(\eta) = u(\pi)\},$$

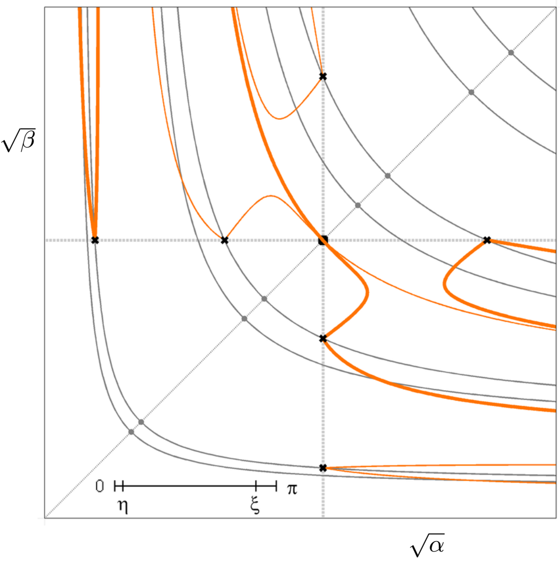
and $L^* : \mathrm{dom}(L^*) \subset L^2(0,\pi) \to W^{-2,2}(0,\pi) : u \mapsto -u''$,

$$\mathrm{dom}(L^*) := \{v \in L^2(0,\pi) : \quad v,\ v' \text{ are absolutely continuous on } (0,\eta),\ (\eta,\xi),\ (\xi,\pi), \\ v'(0) = 0,\ v(\pi) = 0, \\ v(\xi+) = v(\xi-) - v(0), \quad v'(\xi+) = v'(\xi-), \\ v(\eta+) = v(\eta-), \quad v'(\eta+) = v'(\eta-) + v'(\pi)\}.$$



Outline

Asymptotes and Tangent Lines

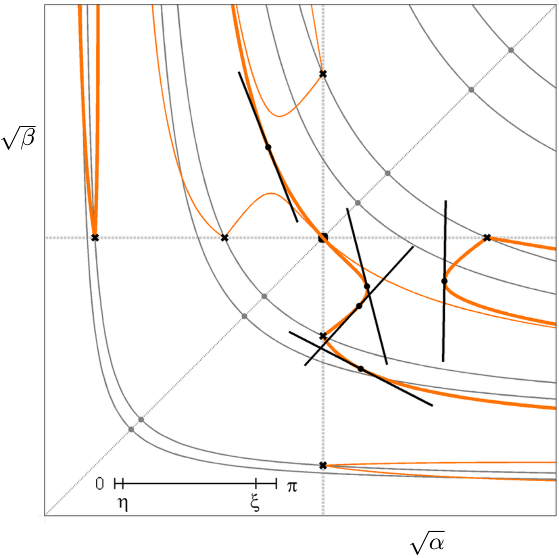


- $\langle |\varphi|, \varphi_{\mathtt{D}} \rangle + \eta \langle \varphi, \varphi_{\mathtt{D}} \rangle = 0,$

  where

$$L\varphi = \left[\alpha\chi\left(\varphi^{+}\right) - \beta\chi\left(\varphi^{-}\right)\right]\varphi,$$
$$L^{*}\varphi_{\mathtt{D}} = \left[\alpha\chi\left(\varphi^{+}\right) - \beta\chi\left(\varphi^{-}\right)\right]\varphi_{\mathtt{D}}.$$

Asymptotes and Tangent Lines



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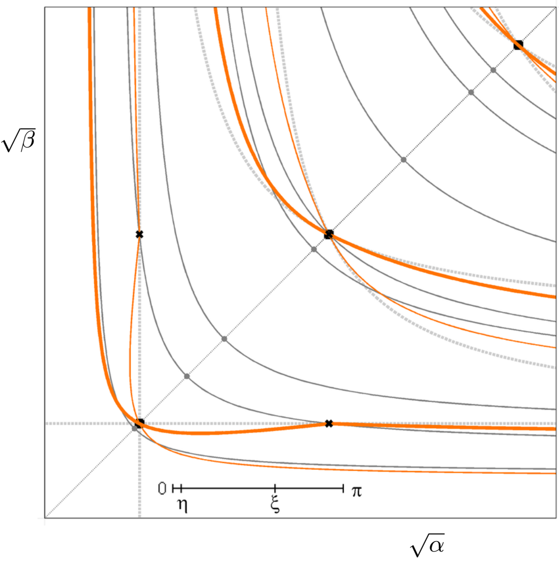
- $\beta'(\alpha) = \dfrac{\eta - 1}{\eta + 1},$

  where

$$\eta = -\frac{\langle |\varphi|, \varphi_{\mathsf{D}} \rangle}{\langle \varphi, \varphi_{\mathsf{D}} \rangle}.$$

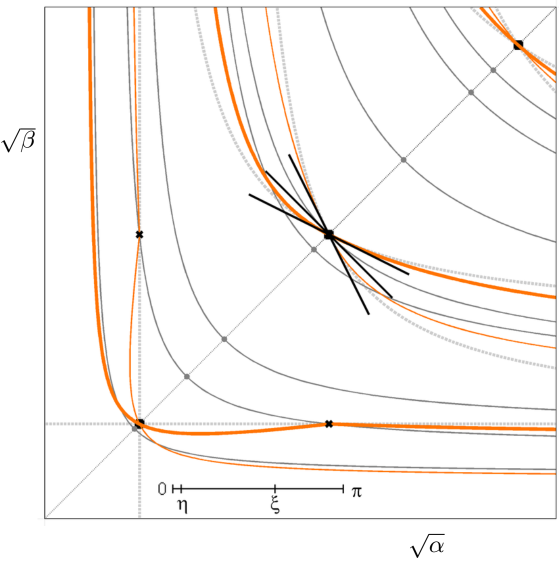
Asymptotes and Tangent Lines



- $\alpha = \beta = \lambda_1^{\mathtt{DN}} = \lambda_2^{\mathtt{P}\xi} =: \lambda,$

$$\begin{aligned} \langle |\varphi| \,, \varphi_{\mathtt{D}} \rangle + \eta \, \langle \varphi, \varphi_{\mathtt{D}} \rangle &= 0, \\ (L - \lambda I)\varphi &= 0, \\ (L - \lambda I)\psi &= \varphi, \\ (L^* - \lambda I)\varphi_{\mathtt{D}} &= 0. \end{aligned}$$

Asymptotes and Tangent Lines



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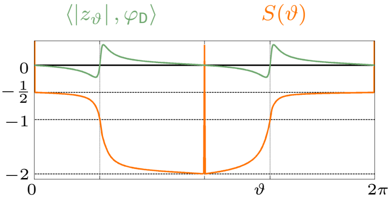
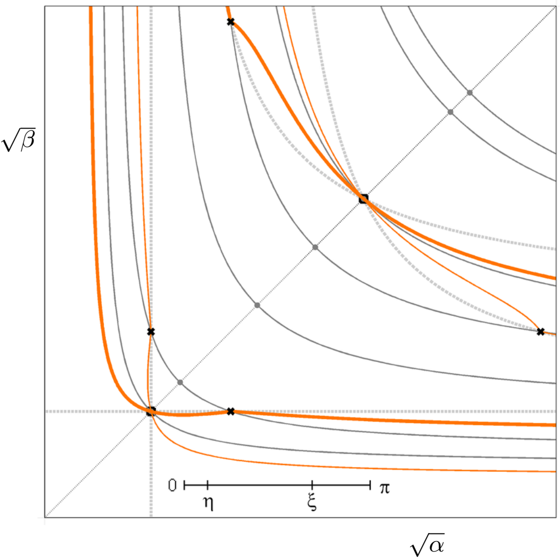
$$\begin{aligned} \langle |\varphi|, \varphi_{\mathtt{D}} \rangle + \eta \langle \varphi, \varphi_{\mathtt{D}} \rangle &= 0, \\ (L - \lambda I)\varphi &= 0, \\ (L - \lambda I)\psi &= \varphi, \\ (L^* - \lambda I)\varphi_{\mathtt{D}} &= 0. \end{aligned}$$

- $S(\vartheta) := \dfrac{\eta - 1}{\eta + 1}, \; \eta = -\dfrac{\langle |z_\vartheta|, \varphi_{\mathtt{D}} \rangle}{\langle z_\vartheta, \varphi_{\mathtt{D}} \rangle},$

$$z_\vartheta(x) = \cos\vartheta \cdot \varphi(x) + \sin\vartheta \cdot \psi(x).$$



Asymptotes and Tangent Lines

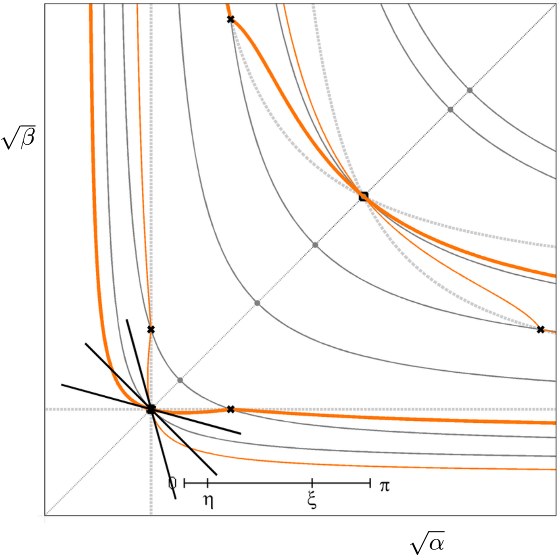


- $\alpha = \beta = \lambda_0^{\mathtt{DN}} = \lambda_1^{\mathtt{P}\eta} =: \lambda,$

$$\langle |\varphi|, \varphi_{\mathtt{D}} \rangle + \eta \langle \varphi, \varphi_{\mathtt{D}} \rangle = 0,$$

$$(L - \lambda I)\varphi = 0, \ (L^* - \lambda I)\varphi_{\mathtt{D}} = 0,$$
$$(L - \lambda I)\psi = \varphi, \ (L^* - \lambda I)\psi_{\mathtt{D}} = \varphi_{\mathtt{D}}.$$

Asymptotes and Tangent Lines



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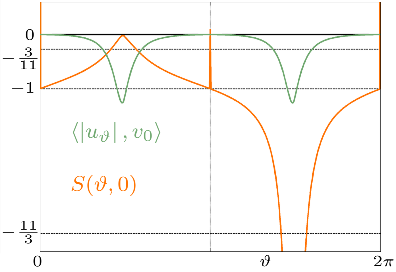
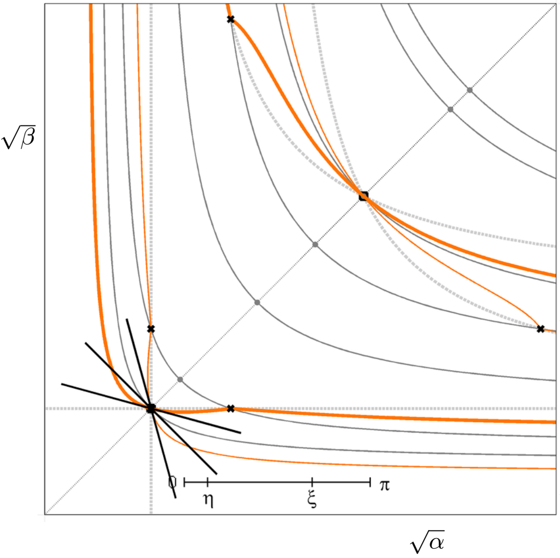
- $S(\vartheta, \omega) := \frac{\eta - 1}{\eta + 1}, \eta = -\frac{\langle |u_\vartheta|, v_\omega \rangle}{\langle u_\vartheta, v_\omega \rangle},$

$$u_\vartheta(x) = \cos\vartheta \cdot \varphi(x) + \sin\vartheta \cdot \psi(x),$$
$$v_\omega(x) = \cos\omega \cdot \varphi_{\mathtt{D}}(x) + \sin\omega \cdot \psi_{\mathtt{D}}(x).$$



Asymptotes and Tangent Lines



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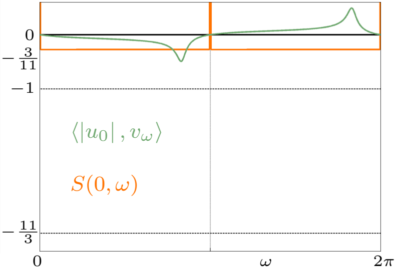
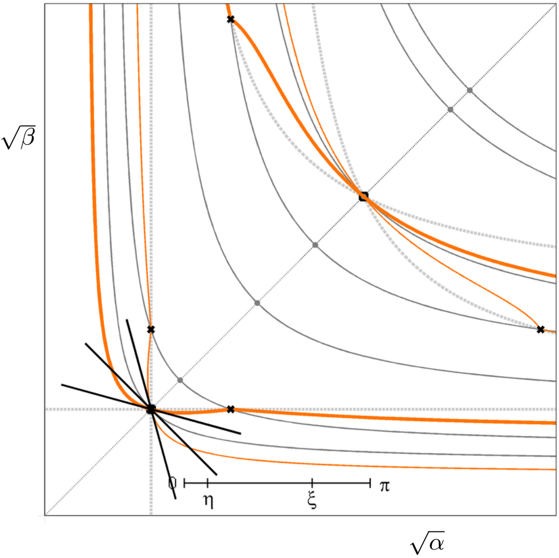


Asymptotes and Tangent Lines



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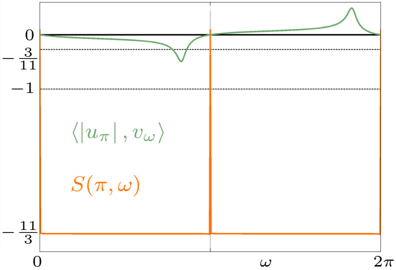
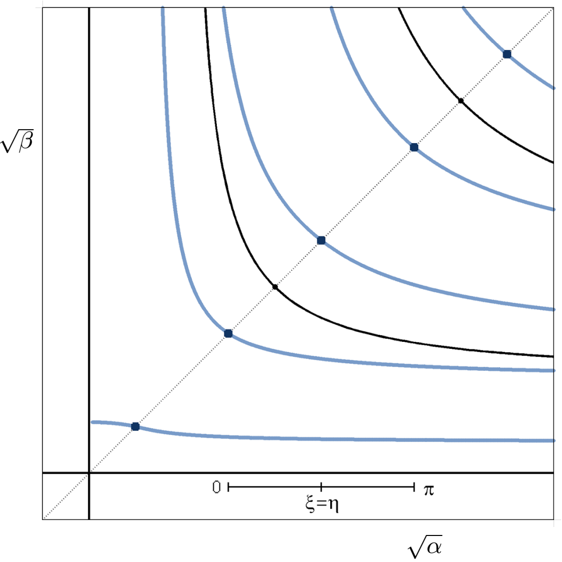


Outline

The Fučík Spectrum of L^* - Numerical Experiment



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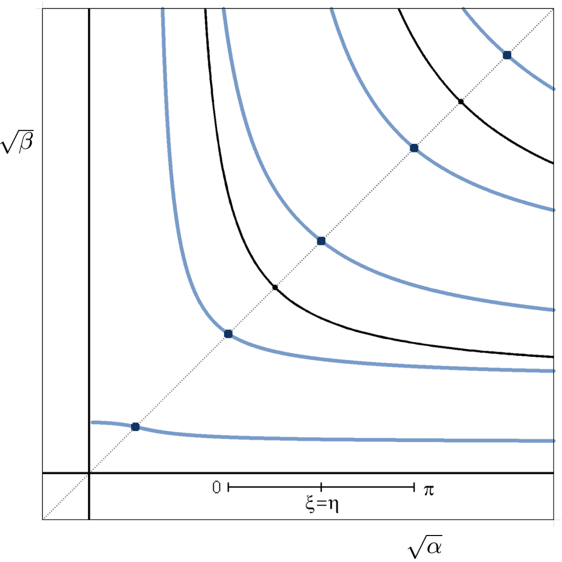
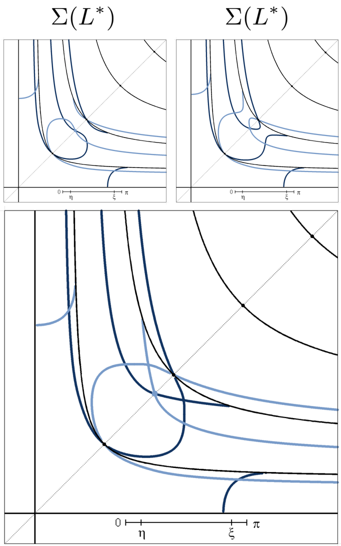
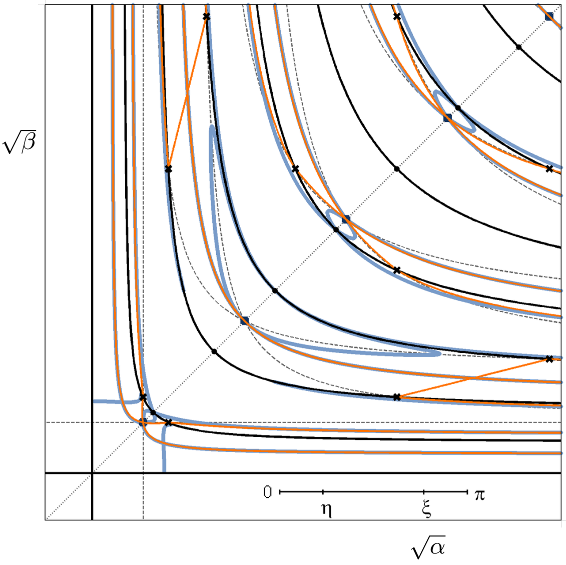
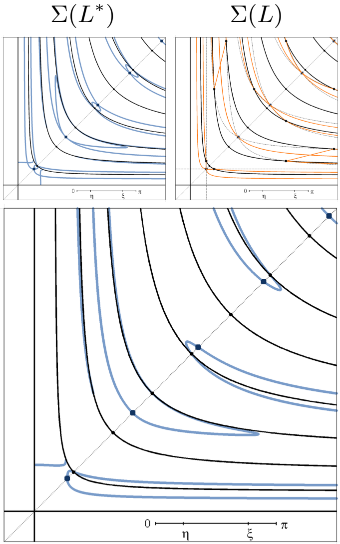




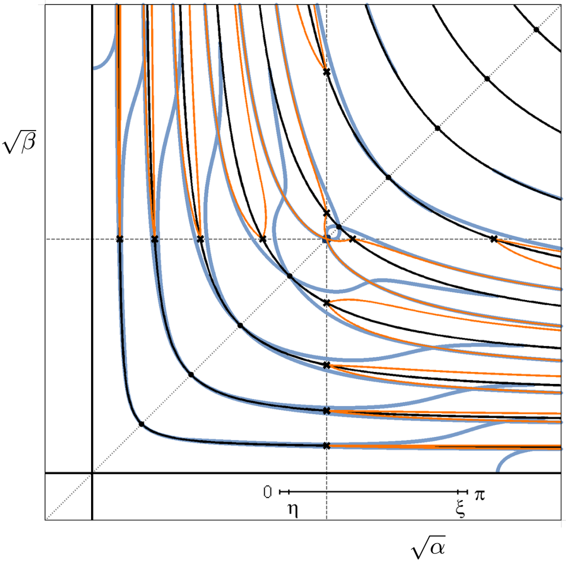
Outline

Connection between the Fučík Spectra of L and L^*

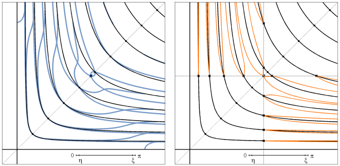


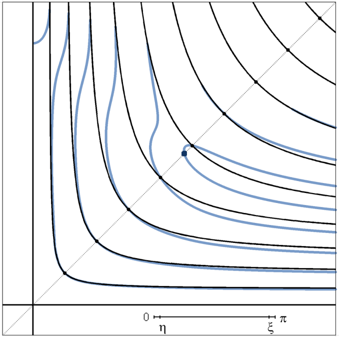


Connection between the Fučík Spectra of L and L^*



$\Sigma(L^*)$ $\Sigma(L)$





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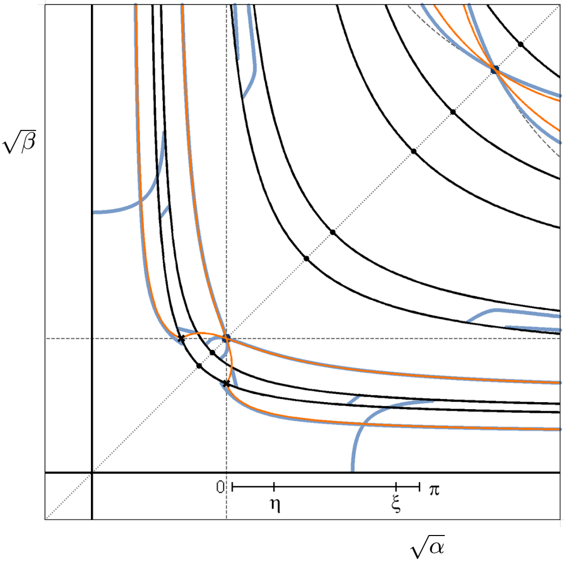

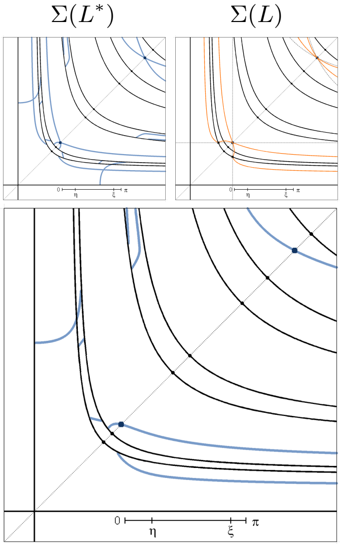

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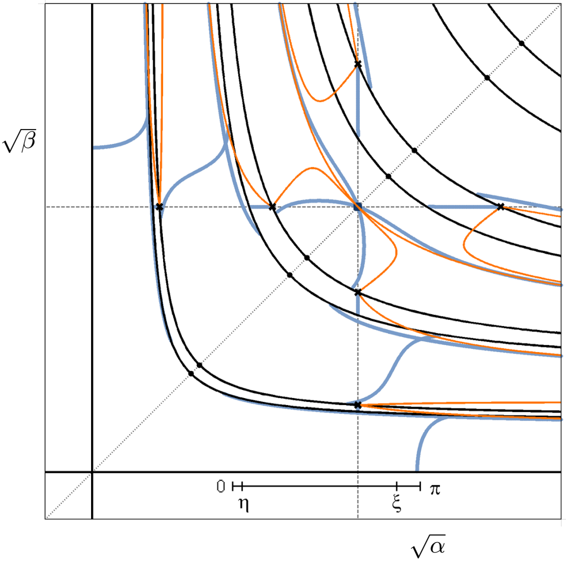

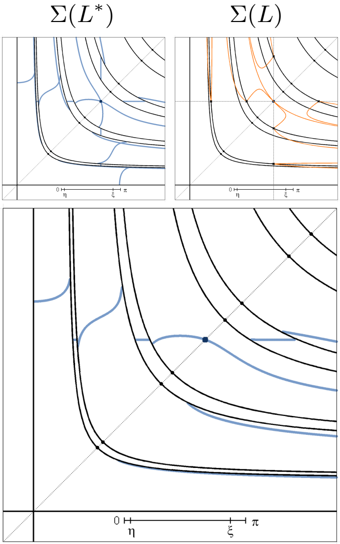

References

S. Fučík
Solvability of Nonlinear Equations and Boundary Value Problems,
D. Reidel Publ. Company, Holland, 1980.

K. Ben-Naoum, C. Fabry and D. Smets
Structure of the Fučík spectrum and existence of solutions for equations with asymmetric nonlinearities,
Proc. Roy. Soc. Edinburgh 131 A (2001), 241–265.

G. Holubová and P. Nečesal
Nontrivial Fučík Spectrum of One Non-Selfadjoint Operator,
to appear in Nonlinear Analysis: Theory, Methods & Applications.

J. Horák and W. Reichel
Analytical and numerical results for the Fučík spectrum of the Laplacian,
J. Comput. Appl. Math. 161 (2003), 313–338.

P. Krejčí
On solvability of equations of the 4th order with jumping nonlinearities,
Časopis pro pěst. mat. 108 (1983), 29–39.

P. Nečesal
The Beam Operator and the Fučík Spectrum,
Proceedings of Equadiff-11 (2005), 303–310.